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Anatomy. — *The Mammalian homologues of the dorsal thalamic nuclei of Reptiles.* By C. U. ARIËNS KAPPERS.

(Communicated at the meeting of March 28, 1942.)

In comparative anatomy it is customary to divide the diencephalon into three parts, the epithalamus (or ganglia habenulae), thalamus proper and hypothalamus. In embryos the thalamus proper is separated from the epithalamus by the sulcus diencephalicus dorsalis, from the hypothalamus by the sulcus diencephalicus ventralis. It may be subdivided by HERRICK's sulcus medius thalami into a dorsal and ventral thalamus. In adult mammals the dorsal thalamus increases considerably and its limit with the ventral thalamus is indicated by the lamina medullaris externa thalami.

The ventral thalamus, frequently called subthalamus (not to be confounded with the hypothalamus), is chiefly concerned with extrapyramidal motility. It is connected with fibres arising from the corpus striatum, specially from its oldest part, the paleo-striatum, the globus pallidus of mammalian anatomy. Although ascending systems also originate from the ventral thalamus e.g. from the nucleus ruber, descending systems prevail in it. On the other hand the dorsal thalamus is largely (not only) intercalated in ascending systems.

Traces of the dorsal thalamic ascending systems first appear in Amphibia (RUBASCHKIN, HERRICK) where, however, they are still poorly developed. Their increase in Reptiles, Birds and Mammals is the most characteristic feature of cerebral evolution.

The dorsal thalamic nuclei of these systems may be divided into neo-cortical and neo-striatal projection centres. The first are most evident in Mammals, where the ever increasing neo-cortex prevails over the less increasing striatum and where, in addition to thalamo-cortical connections, a considerable number of cortico-thalamic fibres occur. In higher Mammals consequently the neo-striatal projection centres of the dorsal thalamus are concealed under an overwhelming mass of neocortical centres.

In Reptiles, however, where a neo-cortex is practically absent, the striatal projection centres of the dorsal thalamus are very distinct. They are chiefly connected with a part of the striatum which, in contradistinction to the older predominant efferent part (the paleo-striatum) is called the *neo-striatum*.

This neo-striatum, still small in Turtles, is well developed in Lizards and Serpents and specially in Crocodiles. — The cortex in these animals is chiefly an olfactory one.

Since neocortical projection systems are hardly developed the neo-striatal projection systems may be far more easily studied in Reptiles than in Mammals. The knowledge of their relations in Reptiles may help us to trace them in Mammals and thus throw some more light on some nuclei to which hitherto little attention has been given.

In the Reptilian dorsal thalamus (fig. 1) EDINGER and DE LANGE already distinguished several nuclei. The most important of them, apart from the lateral geniculate, were indicated by these authors as the *anterior* and *round nuclei* and the *nucleus reuniens*. Laterally to the round nucleus between this and the optic tract an other group of large cells is distinguished by FREDERIKSE as *lateral nucleus*. The two former nuclei and the so-called lateral nucleus are evident in all Reptiles, the *nucleus reuniens* is developed specially in Crocodiles (see also HUBER and CROSBY). In Lizards it is not absent, but its component parts do not or hardly join in the midline, so that they do not form a real midline nucleus but are represented on both sides bulging slightly into the ventricle and touching each other. FREDERIKSE, working on the Lizard, called this non-joining paramedial nucleus the ventral nucleus, a mere topographical indication. As it lies near the midline under the rotundus nucleus it might also be called nucl. *paramedialis subrotundus*.

The names chosen for this and the other nuclei (apart from the lateral geniculate, which

we do not consider here) do not involve any homology with homonymous nuclei in other animals. They were merely chosen to indicate their form or position in the Reptilian brain.

HUBER and CROSBY in their work on the Alligator and SHANKLIN in his description of the Chameleon brain made a further distinction in the so called anterior nucleus, which they divided into a medial small celled group (*nucl. dorso-medialis anterior*) and a lateral large celled group (*nucl. dorso-lateralis anterior*). They furthermore distinguished a cell-group, lying more laterally and ventrally in front of the geniculate as *nucleus ovalis*. This last group was recognized by ADDENS as being the homologue of BELLONCI's nucleus in the Amphibian brain.

In the following discussion we shall not consider the lateral geniculate nucleus (not indicated in fig. 1) which receives some optic fibres and collaterals and which in our opinion is a predecessor of the mammalian ventral geniculate, and BELLONCI's nucleus (N.B. fig. 1) which lies between the olfacto-habenular and optic systems and whose homologue has recently also been described by Miss GILBERT in the human embryo and by ADDENS in the rabbit.

As far as the small celled dorso-medial and large celled dorso-lateral anterior nuclei of HUBER and CROSBY are concerned, comparative study of the Reptilian and lower mammalian thalamus have convinced us that the small celled dorso-medial anterior nucleus is the homologue of the *nucleus paraventricularis parvocellularis anterior* of Mammals described by GURDJIAN, RIOCH and WALKER in Rodents, Carnivores and Primates respectively and by SUZUKI in Xantharpyia.

Both the dorso-medial anterior nucleus of Reptiles and the anterior paraventricular nucleus of Mammals have their greater development in the most frontal part of the dorsal thalamus, both have a more or less semicircular or triangular shape, their basis being turned to the ventricle, and both receive fibres from a system of poorly myelinated fibres that originate in the septum (in Mammals running medially to the fornix), which end partly in this nucleus, partly further down in the hypothalamus. It seems that this bundle is increased by fibres arising from the nucleus. A striking difference between the Reptilian and Mammalian nucleus is that, compared with the remaining thalamus, its relative size in Reptiles is far greater than in Mammals, especially in Primates where it nearly disappears in comparison with the large neocortical nuclei, but where it has been accurately described by WALKER, who also observed that it does not degenerate after cortical lesions and who is also inclined to group this nucleus with those that are intercalated in hypothalamic functions.

The *dorso-lateral anterior nucleus* of Reptiles is in our opinion the homologue to the *parataenial nucleus* of Mammals. In both, Reptiles and Mammals, this nucleus is located immediately against and underneath the taenia thalami (tr. cortico-habenularis + tr. olfacto-habenularis). In both it gives rise to ascending fibres that end in the neostriatum i.e. in the outer part of the Reptilian striatum, which in these animals is not yet divided by an internal capsule into a nucleus lentiformis and a caudate nucleus. In Reptiles these fibres contribute to the dorsal striatal peduncle¹⁾ (P.St.D. fig. 1), in Mammals they form part of the anterior inferior thalamic radiation (P.St.D. and R.Th.I., fig. 2).

Caudally the dorso-lateral anterior or parataenial nucleus of Reptiles extends in a medial direction thus restricting the size of the paraventricular anterior nucleus. Medio-ventrally it borders upon our *nucleus paramedialis subrotundus*, caudally upon the nucleus rotundus from which the larger part of the dorsal striatal peduncle arises.

For the homology of the dorso-lateral anterior nucleus of Reptiles with the parataenial

¹⁾ This peduncle is called "dorsal" in Reptiles in contradistinction to the ventral striatal peduncle which contains the descending fibres arising from the paleo-striatum (or globus pallidus).

nucleus of Mammals¹) we also refer to the fact that NISSL (who in his study on the thalamic centres of the rabbit p. 939 called this nucleus "medialer vorderer dorsaler Kern") and D'HOLLANDER found it intact after cortex extirpation. This is confirmed by WALKER, but while WALKER considers the parataenial nucleus also as being intercalated in hypothalamic connections, J. DROOGLEEVER FORTUYN observed, that although this nucleus may receive some cortical fibres, degeneration in this nucleus was especially evident in his experiment XXI in which, in addition to the lateral neocortex, the neostriatum had also been damaged. In *Xantharpyia* and *Mus rattus norvegicus* the striatal connection of this nucleus is quite evident.

An additional argument for the homology of the nucleus dorso-lateralis anterior of Reptiles and the parataenial of Mammals is the fact that they have the same cytotoxic aspect showing fairly large multipolar cells and that both are characterized by a conspicuous blood supply. Furthermore the fact that the dorso-lateral anterior nucleus of Reptiles touches medio-caudally upon a paramedial subrotundus nucleus, agrees with the observations of D'HOLLANDER and DROOGLEEVER FORTUYN that the parataenial nucleus of the rabbit touches medio-ventrally upon the paramedial cell group located medially to the bundle of Vicq d'Azyr and indicated by NISSL as "medialer vorderer ventraler Kern", which in Mammals — as in Crocodiles — caudally joins with its fellow of the other side in the *nucleus reuniens* (R. fig. 2).

While the nucleus parvocellularis paraventricularis anterior is intercalated in septo-hypothalamic, probably autonomic, systems and the parataenial nucleus has neo-striatal connections, it is difficult to state the function of the *mammalian paramesial nucleus* (and of its caudal junction the *nucleus reuniens*). D'HOLLANDER '13 and GURDJIAN '27 found connections with the inferior thalamic radiation in the rabbit. DROOGLEEVER FORTUYN asserts that it has cortical connections in this animal. MÜNZER and WIENER (who indicated this nucleus by the name of "ventral arcuate nucleus") denied its cortical character. WALKER does not mention this nucleus in the Macaque. It may, however, be included in his "massa grisea centralis" (l.c. '38, p. 37). As far as concerns the middle part of the nucleus reuniens of D'HOLLANDER, the so called central nucleus, FORTUYN as well as LE GROS CLARK admit that this has no cortical connections and BODIAN recently stated that it has striatal connections. Paraventricular fibres seem to connect it with the hypothalamus.

This makes it probable that the medial part of the paramesialis and the central nucleus of the massa reuniens is homologous to the paramesialis and reuniens nucleus of Reptiles.

This conclusion is supported by the fact that in the area of the central nucleus (2 of D'HOLLANDER's *noyau réunissant*) and immediately behind it, a large number of decussating fibres of caudal origin occur, that may be homologous to the decussating fibres of tectal and subtectal origin ending in the nucleus reuniens of Reptiles.

According to GLORIEUX the fibres²), decussating behind the central nucleus of Mammals, arise at least partly from the medial geniculate nucleus. This makes it probable that they are homologous to the crossed fibres which arise in the nucl. commissurae transversal which is homologous in the nucleus geniculatus C of Mammals or nucleus subgeniculatus (a real medial geniculate nucleus does not yet occur in Reptiles, where a cortical projection of the auditory sense fails).

¹) As may be expected this nucleus, as well as the paraventric anterior, is especially distinct in lower mammals *Didelphys* (CHU, '32), *Armadillo* (PAPEZ, '32); rat (GURDJIAN, '27), *Xantharpyia* (SUZUKI, '36), dog and cat (RIOCH, '29) and INGRAM HANNETT and RANSON, '32). In Primates it has a much smaller relative size (C. VOGT, '09), FRIEDEMANN, '12), (LE GROS CLARK, '30 and WALKER, '38).

²) Not to be confounded with the commissural fibres between the anteroventral or other bilateral nuclei.

As to the homologue(s) of the reptilian *rotundus*, in various papers we expressed as our opinion that this had to be sought among the medial nuclei of Mammals, especially in the nucleus medialis b of VON MONAKOW, which according to this author corresponds to the centre médian of LUYSS (according to RIOCH only its caudal part corresponds to the centre médian) and possibly in the most medial division of the ventral nucleus. Since then analogous opinions have been expressed by several other authors (FOIX and NICOLESCO, PAPEZ, KODAMA). More recent researches seem to confirm this conception and throw more light on the topographic changes of these thalamo-striatal centres in Mammals and their functional character.

Degenerative observations recorded in literature concerning the striatal projection centres in the mammalian dorsal thalamus being scanty, we shall first of all consider the topographic relations of the nucleus rotundus of Reptiles (fig. 1) and see which nucleus (or nuclei) of the mammalian thalamus has (have) analogous relations.

Frontally the rotundus nucleus of Reptiles borders upon the anterior parvocellular paraventricular or dorsomedial anterior nucleus and upon the parataenial or dorso-lateral anterior nucleus. Dorsally to it arises the fasc. retroflexus which in its further course passes caudo-laterally along this nucleus. Medio-ventrally the rotundus borders upon the paramedialis and reuniens. Caudally the nucleus rotundus flattens out laterally. Thus flattened it diminishes, gradually extending underneath the pretectal (prebigeminal) region unto the level of the posterior commissure of the midbrain. Here the lateral part of the nucleus is more and more occupied, and finally replaced by the strongly myelinated fibres of the tecto-thalamic tract, a bundle first described in birds by EDINGER and WALLENBERG and in Reptiles by HUBER, CROSBY and SHANKLIN.

This tract, arising from the caudo-lateral part of the mesencephalon at the junction of the tectum (corp. bigemina anteriora) and the corp. bigemina posteriora ends, in the rotundus nucleus as well as in the parataenial nucleus. Somewhat finer fibres probably arising in the nucl. commissurae transversae (= nucl. subgeniculatus of Mammals) join the ventral side of the coarse fibred tecto-thalamic tract and seem to end in the paramedial (or subrotundus) nucleus and in Crocodiles in its confluence: the reuniens nucleus. Behind the reuniens (which lies in between the nuclei rotundi) a large number of poorly myelinated paraventricular fibres occur, most of which continue in the posterior paraventricular bundle of SCHÜTZ, while others may connect the dorsal thalamus with the hypothalamus (see fig. 1).

Turning to the Mammalian thalamus (fig. 2) we shall first consider the possibility if the nucleus rotundus of Reptiles may be included in the caudal part of the mammalian parataenial that extends further backward than its reptilian homologue and in which RIOCH distinguished a caudal part from the oral part by its more closely packed cells. In *Xantharpyia* this caudal division is conspicuous also by its round shape. Considering the close topographical relation between the nucleus dorso-lateralis anterior or parataenialis of Reptiles and the rotundus of these animals the possibility that also in Mammals these two nuclei lie closely against each other is not to be excluded.

It is, however, also possible that in Mammals the rotundus nucleus is pressed ventrally by the newly formed neocortical nuclei, by the triad of anterior nuclei (especially by the nucl. antero-ventralis and antero-medialis) and by the dorso-medial nucleus which do not exist in Reptiles. This is the more likely so as also the paramedial nucleus and its junction the reuniens to which the rotundus nucleus is ventrally attached in Reptiles, are pushed ventrally in mammals by these neo-cortical nuclei.

If this holds good also for the mammalian homologue of the reptilian rotundus, as it probably does, we have to look for it behind and below the anterior and dorso-medial nuclei lying on the paramesial and reuniens nuclei, and bordering caudally upon the fasc. retroflexus.

The bilateral nuclei located underneath the dorso-medial nucleus are the centre médian

(caudally bordering upon the parafascicular nucleus), the nucleus submedius (or ventro-medialis¹), furthermore the paracentral, the centralis lateralis and the ventralis postero-lateralis (= the nucleus of the lemniscus medialis) and ventralis postero-medialis sive arcuatus (the nucleus of the trigeminal lemniscus).

Recent investigations, among which MORRISON's, LE GROS CLARK's, WALKER's, DROOGLEEVER FORTUYN's and PAPEZ' experiments, make it very unlikely that the lemniscal nuclei in addition to their cortical projections should have striatal projections, decortication producing a profound atrophy of these nuclei. The only nuclei which after decortication remain entirely or largely intact, apart from the parataenial, parvocellular paraventricular and central nucleus are the parafascicularis, the centre médian and submedius. In addition to these WALKER mentions the nucleus ventralis anterior.

WALKER is inclined to consider the ventralis anterior (which, if it has a homologue in the Reptilian brain might be represented by the lateral part of the rotundus nucleus or by FREDERIKSE's nucleus lateralis caudally to the nucl. dorso-lateralis anterior of Reptiles) and the centre médian as striatal nuclei, the first named one being especially neostriatal (i.e. connected with caudate and putamen), the last as a rather paleostriatal nucleus. LE GROS CLARK considers the submedius as the homologue of the rotundus. In view of the fact that the mammalian neostriatum is much larger than that of the Reptiles, it may be possible that the centre médian, considered by HUBER and CROSBY ('26) and RIOCH ('31) as the homologue of the rotundus as well as the submedius, and also the parafascicularis that immediately borders upon these nuclei and whose cells resemble those of the centre médian have to be considered as being derived from the rotundus of Reptiles. The topographic relation of the centre médian and submedius to the centralis, their close proximity to the parafascicularis, the fact that the fasc. retroflexus runs immediately along the parafascicularis (and subparafascicularis) strongly reminds us of the topographic relation of the rotundus to this bundle in Reptiles.

The hypothesis that the centre médian (the caudal part of VON MONAKOW's nucl. medialis b), submedius and parafascicularis are the mammalian homologues of the rotundus in striatal projections, at the same time may shed some more light on the physiological character of this system.

The lamina medullaris interna in which the submedius and centre médian are located contains tectal projection fibres, some of which end in the centre médian (LE GROS CLARK). These fibres may include those of GLORIEUX' *commissura principalis thalami* i.e. the commissural fibres that decussate immediately behind and on the level of the submedius nuclei. As stated above these fibres arise at least partly from the tectum and from the region of the medial geniculate nucleus, i.e. from a region closely related to the region from which the tecto-thalamic bundle of reptiles (and birds) arises. This supports the supposition that the submedius and the centre médian represent the rotundus and that also in mammals their striatal projection has to do with a tectal system, serving the coordination of reflex action. In addition two other connections should be considered: a spinal-trigeminal projection and a connection with the cerebellum. According to WALLENBERG the centre médian of the rabbit receives fibres from the spinal trigeminal nucleus (Tr. spin. V thal. fig. 1 and 2). Although this is doubted for the Macaque by LE GROS CLARK and WALKER, the former ('36 p. 381) admits that, while in the Macaque the majority of the fibres end in the arcuate nucleus (WALKER's nucl. ventralis postero-medialis) some fibres of the spinal Vth projection which run through the lamina medullaris interna (see also DROOGLEEVER FORTUYN p. 68—70) may end in the centre

¹) It is preferable to indicate this group as nucl. submedius, as it lies directly under the nucl. dorso-medialis. The name ventro-medialis (chosen to indicate its position under the dorso-medialis) easily leads to confusion with the nucleus ventralis internus (sometimes called ventralis medialis), which lies on a more ventral level laterally to and above the bundle of vicq d'Azyr (not indicated in our fig. 2). This confusion has led to some erroneous descriptions of the nucl. submedius.

to C. VOGT and WALKER the nucleus ventralis anterior of mammals has also neo-striatal connections. We do not know if this nucleus is related to the rotundus system. Its position seems too lateral for such a comparison. If not included in the lateral part of the nucl. rotundus of reptiles it might be represented by FREDERIKSE's lateral nucleus (l.c. fig. 13 and 14), located behind the dorso-lateralis anterior, between the rotundus and the lateral geniculate. The connections of this nucleus in reptiles are rather obscure.

It seems that the nucl. ventralis anterior of Mammals receives cerebellar impulses. Although according to WALKER the brachium conjunctivum ends in the prelemniscal part of the nucleus ventro-lateralis (ventro-later. p. cer. fig. 2) located immediately behind the ventralis anterior (fig. 2), in his pictures (l.c. '37, fig. 5 No. 77 and l.c. '38, fig. 18 No. 77) several fibres of the brachium are seen proceeding more frontally and entering the ventralis anterior. Consequently this nucleus may carry cerebellar impulses to the striatum. Moreover GEREBTZOFF states that most fibres of the brachium end in a more dorsally lying group of cells designated as nucl. magnocellularis thalami¹⁾ while other fibres of it — together with spinal trigeminus projections — terminate in the parafascicular nucleus. As also RANSON and INGRAM mention the parafascicular nucleus as a terminus of brachium fibres this nucleus would combine the functions of a spinal trigeminal and cerebellar centre.

Conclusions.

Summing up we are inclined to homologize the dorsomedial anterior nucleus of Reptiles with the paraventricularis anterior of Mammals, the dorso-lateral anterior with the parataenialis and the paramesialis (subrotundus) and reuniens with the paramesialis and reuniens of mammals. The rotundus nucleus of Reptiles may be represented by the centre médian (caudal part of nucl. medialis b. v. MONAKOW), submedius and parafascicularis. Some of these nuclei (the paraventricularis parvocellularis anterior and reuniens) are instrumental as hypothalamic relays, others (the parataenial, submedius, centre médian and parafascicular) may have striatal connections. In Reptiles all these nuclei lie closely together, in Mammals the triad of anterior neocortical nuclei and the nucleus dorso-medialis have pushed its component parts apart, so that only the paraventricularis parvocellularis anterior and parataenialis keep their original position in the frontal pole of the thalamus. The same would hold good for the ventralis anterior of Mammals if this nucleus corresponds to the nucleus lateralis of Reptiles (whose connections hitherto are unknown). The paramesial and reuniens are pushed ventrally by the antero-ventralis (ant. vent.) and antero-medialis (ant. m.), neocortical nuclei not present in reptiles.

Although the centre médian, the submedius and parafascicular nuclei are pushed downward and backward by the neocortical dorsomedial nucleus (neither present in reptiles) they retain the topographical relation to the fasciculus retroflexus shown by the rotundus complex of reptiles. The tectal, cerebellar and some spinal trigeminus projections terminating in this complex may transmit tonus regulating influences to the striatum probably for the movements of the head and neck especially.

We want to add that the striatal character of the mammalian nuclei mentioned does not necessarily exclude that some of them may have additional cortical connections, especially descending (inhibiting) ones, since also the striatum itself in mammals receives some cortico-fugal fibres.

The diagrams added to this paper are chiefly intended to demonstrate the topography of the nuclei discussed. The lateral nuclei of the thalamus are only partly indicated. It is obviously impossible to project in one plane all thalamic nuclei in their actual relations, since the projection of the more peripheral ones would entirely cover those located more medially, some of which also partly overlap.

For lack of space the bibliographic references to this paper could not be printed.

¹⁾ Included in the nucl. centralis lateralis of other authors.

Meteorology.— *A remarkable 8-year period in air-temperatures.* By P. TETRODE. (Communicated by Prof. E. VAN EVERDINGEN.)

(Communicated at the meeting of March 28, 1942.)

According to Mrs. MAUNDER¹⁾ and to SANFORD²⁾, it looks as if among all planets the Earth and Venus are foremost to affect sunspotnumbers. Mrs. MAUNDER showed, that from 1889—1901 947 groups of sunspots came into view around the east limb of the sun or were formed close to it, while only 777 groups passed around the west limb or were dissolved close to this one.

SANFORD found that for the ten days with superior and with inferior conjunctions Venus—Earth, which occurred during 1917—1932 (1917 for the first time daily sunspotnumbers were published by the International Astronomical Union), sunspotnumbers averaged 71.4 and 39.6 respectively. For the 20 times five days beginning two days before these conjunctions and ending therefore two days after these numbers averaged 68.1 and 38.5 respectively, while for the 20 pentades at whose third day the heliocentric longitude of both these planets differed exactly by 90° the mean number amounted to 43.0.

A few years ago long periods in the sunspotnumber and other were applied in long range forecasting in this country — among these one of $8\frac{1}{3}$ years³⁾.

Hence the thought occurred to me to examine the question, whether the fairly stable synodical period, which between two successive conjunctions has an average length of 583.92 days, might be traced in air temperatures on the earth. The result was not very satisfactory. Because of the opposite sign temperature deviations from the normal caused by influences of this kind might assume in different seasons, I sought the least multiple of this period forming a full number of years, i.e. five times the period, or 8 years minus 2.3 days. Temperature records of the stations available for this purpose as a rule do not reach back materially more than 160 years; but 20 periods of 8 years might be considered sufficient to eliminate accidental deviations, and in this fivefold period difficulties originating from the seasons are removed as well as possible.

When a result was obtained for the longest record available in this country, I looked for a few series of temperature records dating far enough back elsewhere, in order to test the reality of this period. Besides the series Zwanenburg—Utrecht—de Bilt we found Prague (homogeneous, unchanged site and available as well from 1780 up to 1940) and New Haven on the Atlantic border some 100 km to the North of New York, the only series available outside Europe (1780—1930). For each of these series as a whole for the 8-year period 32 overlapping yearly means were computed, for certain reasons contrary to custom commencing with February, May etc. Moreover these means were computed also from 1852, the epoch at which generally modern observations may be taken to begin for the three stations mentioned and for Charleston, some 1000 km SW of New York on

1) Mrs. A. S. D. MAUNDER, An apparent influence of the Earth on the number of areas of Sun-Spots in the cycle 1889—1901, Monthly notices Roy. Astron. Soc. 1907 May.

2) F. SANFORD, Influence of planetary configurations upon the frequency of visible sunspots. Smithsonian. Misc. Coll. 95, No. 11, 1936.

3) e.g.: E. VAN EVERDINGEN, Regenval en verdamping sinds 1 Januari 1935 en de water-voorraad in den bodem. Water, No. 13, 1938, 1 Juli;

P. TETRODE, Voorwaarden voor belangrijke invloeden van de zonnevlekkenperiode op ons weer en een voorspelling op zeer langen termijn. Hemel en Dampkring 37, p. 225 (1939). Temperatuurprognose voor November, *ibid.* p. 412. Verwachtingen voor dezen winter, *ibid.* 38, p. 90 (1940).

TABLE I.

Yearly means of deviations from normal of the temperatures in the 8-year period in hundred of centigrades e.g. for the first three series 1st. column corresponds with Febr. 1780—Jan. 1781, Febr. 1788—Jan. 1789,; 2nd. with May 1780—April 1781, May 1788—April 1789,, etc. ...

	Netherlands (1780-1939)								Prague (1780-1939) . .								New Haven (1780-1930)								Netherlands (1852-1939)								Prague (1852-1939) . .								New Haven (1852-1930)								Charleston (1852-1930) .								Batavia (1866-1930) . .																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																	
	-11	-18	-22	-32	-20	0	5	11	8	-8	-14	-11	-14	-5	-6	-21	-10	-21	-9	5	17	31	21	25	22	18	19	19	13	6	10	3																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																										</

TABLE II.

Two quarterly means with the largest positive (P) and negative (N) deviations from normal in centigrades in the 8-year period where I corresponds with Febr.—April 1932, XXIV with Nov. 1937—Jan. 1938, XXV with Febr.—April 1938, etc.

	Netherlands				Prague				New Haven ¹⁾ 2)				Charleston ²⁾				Batavia ³⁾			
	P	N	P	N	P	N	P	N	P	N	P	N	P	N	P	N	P	N	P	N
1780—1939 .	0.59 (XXIV) 0.40 (XXV)	-0.55 (IV) -0.43 (XIX)	0.81 (XXIV) 0.58 (XXV)	-0.58 (IV) -0.58 (V)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)	0.88 (XXIV) 0.44 (XXV)	-0.50 (XXIX) -0.47 (XXVI)
1852—1939 .	0.73 (IX) 0.71 (XXIV)	-0.58 (XIX) -0.57 (XVI)	0.72 (XXIV) 0.63 (XVII)	-0.66 (XVI) -0.60 (XIX)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)	1.11 (XXIV) 0.56 (IX)	-0.89 (XXIX) -0.45 (XXXI)

¹⁾ 1780—1930, ²⁾ only until 1930 inclusive, ³⁾ 1866—1930.

the Atlantic border, where the uninterrupted series reaches less far back, — as contrasted with the interrupted one, which is the oldest outside Europe — and also for Batavia, where recordings do not start until 1866⁴). (See graph. and table I.) In a separate table II for each station the four quarters of the 8-year period are indicated, which show the two largest positive and negative deviations from normal. The probable errors of a single result were computed for the calendar years 1873—1923 (table III); the reason for

TABLE III.
Probable errors for the calendar-years 1873—1923 in centigrades.

Netherlands	0.39
Prague	0.41
New Haven	0.50
Charleston U.S.A.	0.42
Batavia	0.18

this restriction is the lack of homogeneity of earlier American records and the apparent variations of very long duration in air temperatures, endangering the reality of the probable error (e.g. average temperature at Prague 1800—1824 9°94; 1875—1899 9°06). Lastly for the same reasons and with a view to long-range forecasting the numbers of positive and negative changes for one year at the warmest phase in this period were computed — not the numbers of positive and negative deviations from the normal (table IV).

TABLE IV.
Numbers of positive and negative changes for one year at the warmest phase in the 8-year period.

		Changes	
		positive	negative
Zwanenburg—De Bilt	from 1780	13	7
Prague	" "	15	4
New Haven	" "	14	3
Charleston	" 1738	14	2
(1760—1822 failing)			

The salient feature which emerges from these tables and graphs is the stability of the peak of the curves. Notwithstanding the rather slight amplitudes as compared with the probable errors, this stability seems to be the conclusive evidence for the meteorological reality of this cycle — especially of this cyclic return at all stations of the peak in the mean temperature for the same three months out of the 96 of this period. This peak moreover almost everywhere shows a larger deviation from the normal temperature than the coldest quarters and years of the cycle.

At this point it was appropriate to look, whether a period of this length had already attracted attention in former times.

According to WOEIKOF, for Stockholm this cycle is the most important of many others which he investigated. In his opinion the real length of this cycle amounts to 7.9 years,

⁴) CH. M. A. HARTMAN, Het Klimaat van Nederland B. Luchttemperatuur, Meded. en Verh. K.N.M.I., No. 24.

HLAVÁČ, Meteor. Zeitschr. Vol. 75, p. 267 (1940); World weather records, Smithsonian. Misc. coll., Vol. 79 and 90.

while he hints at a subdivision in two parts which is not absolutely incompatible with our result. He found that from 1825/6 onwards every eighth winter (Dec.—Febr.) there was on the average $2^{\circ}4$ too warm and with only one exception of $-1^{\circ}2$ centigrade *all* these winters were at least $1^{\circ}4$ too warm⁵⁾. Afterwards PETERSSON showed the same for Berlin and found sometimes a corresponding feature for deep sea currents in some Norwegian fjords⁶⁾. These results correspond with our peak quarter (No. XXIV) Nov.

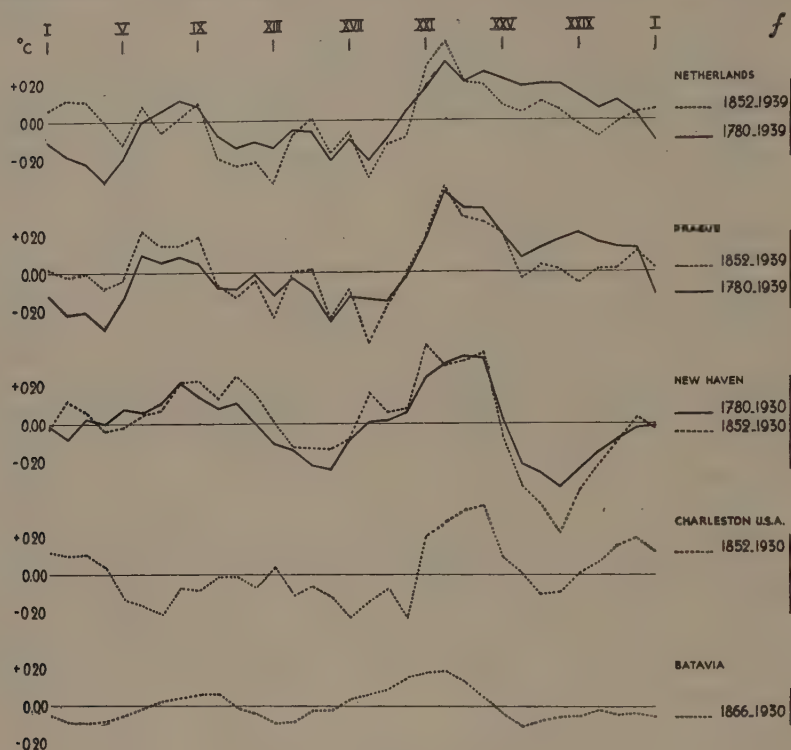


Fig. 1.

Yearly means of deviations from normal temperatures in the eight-year period; e.g. I. — corresponds with Febr. 1780—Jan. 1781, Febr. 1788—Jan. 1789, II. — with May 1780—April 1781, May 1788—April 1789,, etc.

1937—Jan. 1938. MAURER found, that in Switzerland from the beginning of the records in 1816 for the Nov.—Jan. quarter the 8-year period was very well marked in atmospheric pressure with a range of more than 6 mm Hg. since the beginning in 1866 of modern recordings, and that the same applied for Brussels though in a minor degree. For Switzerland the maxima are rather stable and occurred 1818/9.....1912/3⁷⁾. Among numerous other and some more important ones, BEVERIDGE finds a period of 8.05 years in the mean cornprices of some 50 localities in Mid- and Western Europe, rather well marked for about 200 years and gaining ground since⁸⁾. BIGELOW thought to detect traces of this period —

⁵⁾ A. WOEIFOF, Perioden in der Temperatur in Stockholm. Meteor. Zeitschr. **41**, p. 433 (1906).

⁶⁾ O. PETERSSON, Etudes sur les mouvements dans l'air et dans la mer, Hydrografisk. Biol. Kommiss. Skrifter. Vol. 6.

⁷⁾ J. MAURER, Die periodische Wiederkehr hohen Luftdruckstandes im Winter des Alpengebiets, Meteor. Zeitschr. **53**, p. 95 (1918).

⁸⁾ Sir WILLIAM BEVERIDGE, Weather and Harvest cycles, Economic Journal, p. 429 (1921).

the only longer one — in mean atmospheric pressure records over large areas for the calendar years 1873—1899 both in the U.S.A. (especially the plains) and S.W. Europe. The amplitudes are small, a few millimetres at most. As in Switzerland for Nov.—Jan., the maxima are the more stable features, while they appear rather regularly with not more than one year difference⁹⁾. Finally the peaks of my period of $8\frac{1}{24}$ years in November temperatures in this country already mentioned above occurred in the years $1744/5 + n \times 8$, $1849/50 + n \times 8$ — this too not without accordance with the peak in the fivefold synodical period.

Even PLINIUS mentions already a cycle of 8 years in these words: „Indicandum et illud, tempestates ipsas suas ardores habere quadrinis annis, et easdem non magna differentia reverti ratione solis octonis vero augeri easdem, centesima revolvente se luna”¹⁰⁾.

Though nowhere any mention is made of the synodical period conjunction Venus—Earth, these investigations nevertheless reinforce what has been shown above: the existence of a marked and stable cycle in air temperatures, which takes its conspicuousness from its coincidence with the other period, an astronomical one with in turn a conspicuousness of its own: its reflection in sunspot numbers as shown by SANFORD.

I am greatly indebted for the many clarifying remarks Professor VAN EVERDINGEN made to me when I talked the subject of this paper over with him.

⁹⁾ F. H. BIGELOW, Report Chief Weather Bureau 1900—1901, Vol. 2, p. 1004/5.

¹⁰⁾ G. PLINIUS Secundus, Naturalis historia, Liber XVIII, cap. 25.

(Communicated at the meeting of March 28, 1942.)

Der auf S. 137 der trefflichen Einleitung in die Philosophie der Mathematik von Dr. E. W. BETH¹⁾ enthaltene Hinweis auf die vor einigen Jahren von FREUDENTHAL und HEYTING in *Compositio Mathematica*²⁾ über die intuitionistische Deutung logischer Formeln geführte Diskussion veranlasst mich zur Veröffentlichung der nachstehenden deutschen Uebersetzung eines Fragmentes meines am 30. März 1936 an Herrn HEYTING gerichteten Briefes (dem Absatz 4. der zitierten Heytingschen Erörterung schon Rechnung trägt):

„Eine beliebige stetige Funktion kann einen *Punktkern* eines *topologischen Raumes* (wie ich in meinem Aufsatz „*Intuitionistische Einführung des Dimensionsbegriffes*“ beschrieben habe³⁾) stetiger Funktionen darstellen. Die durch ein Gesetz bestimmten Funktionen sind in diesem topologischen Raume die „scharfen“ Punktkerne, genau so wie die Zahlen $\frac{1}{2}$, π , e usw. als „scharfe“, d. h. durch ein Gesetz bestimmte, Punktkerne des Zahlenkontinuums erscheinen. Einen sehr einfachen topologischen Raum stetiger Funktionen des Einheitsintervalles bilden z.B. die Funktionen $y = \sum \pm \frac{x^n}{n!}$, wo der Reihe nach für jede natürliche Zahl n das entsprechende Vorzeichen frei gewählt wird.

In meinen Schriften tritt das obenstehende vielleicht nicht deutlich hervor (bei der ersten Einführung des intuitionistischen Funktionsbegriffes beschränkte ich mich ja auf durch ein Gesetz bestimmte Funktionen⁴⁾); jedenfalls habe ich in meinen Vorlesungen und Vorträgen seit geraumer Zeit betont, dass eine *beliebige* stetige Funktion genau so „im freien Werden“ entsteht wie ein *beliebiger* Punkt des Kontinuums⁵⁾).

Anschliessend mache ich folgende Bemerkungen zu den meinen Abhandlungen zur Begründung der intuitionistischen Mathematik zugrunde liegenden Definitionen⁶⁾:

1. Zur Erklärung einer Menge M gehört nach der Mengendefinition⁷⁾ eine Fundamentalreihe $\alpha(M)$ von Zeichenreihen, welche der Fundamentalreihe der in einer bestimm-

1) Dr. E. W. BETH, *Inleiding tot de wijsbegeerte der wiskunde*, Nijmegen-Utrecht, Dekker & Van de Vegt, 1940.

2) Bd. 4, S. 112—118 (1936).

3) Proc. Kon. Akad. v. Wetensch. XXIX (1926), S. 855. Gemeint werden die dortigen katalogisiert-kompakten Spezies (der vorliegenden Uebersetzung beigegebene Fussnote).

4) Vgl. *Begründung der Funktionenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*, Verhandelingen Kon. Akad. v. Wetensch., I. Sektion, Bd. XIII, No. 2 (1923); *Ueber die Zulassung unendlicher Werte für den Funktionsbegriff*, Proc. Kon. Akad. v. Wetensch. XXVII (1924), S. 248; *Ueber Definitionsbereiche von Funktionen*, Mathem. Annalen, Bd. 97, S. 60—75 (1926) (der vorliegenden Uebersetzung beigegebene Fussnote).

5) Allerdings braucht man zur Repräsentierung der *Gesamtheit* der vollen Funktionen des Einheitskontinuums eine non-finite Menge, während man für die Repräsentierung der *Gesamtheit* der Punktkerne des Einheitskontinuums mit einer finiten Menge auskommt (der vorliegenden Uebersetzung beigegebene Fussnote).

6) Mathem. Annalen, Bd. 93, S. 244 sqq. (1925).

7) Mathem. Annalen, Bd. 93, S. 244, 245.

ten Weise abgezählten endlichen Wahlfolgen eineindeutig zugeordnet ist, und zwar ist nach der Mengendefinition dieses $\alpha(M)$ für M von vornherein festgelegt. Man könnte nun die Frage aufwerfen, ob nicht auch die Betrachtung „schwebender Mengen“ M_σ , für welche die entsprechenden $\alpha(M_\sigma)$ sich im „freien Werden“ befinden, mathematische Fruchtbarkeit besäße. In den Fällen, für welche diese Frage bejahend zu beantworten wäre, werden sich meiner Ueberzeugung nach die betreffenden M_σ immer als mathematische Entitäten oder Spezies herausstellen. Ein einfacher derartiger Fall tritt z.B. ein für diejenigen M_σ , deren $\alpha(M_\sigma)$ die Elemente einer gegebenen Menge M darstellen. Diese M_σ sind Teilspezies einer aus M herleitbaren Menge M_1 , mit welcher ihre Vereinigung identisch ist.

2. Die etwas kurz gehaltene Fussnote zur Definition des Mengenelementes⁸⁾ dürfte in der folgenden ausführlicheren Fassung an Deutlichkeit gewinnen:

Die Fortsetzbarkeitsfreiheit einer von einer unbegrenzten Wahlfolge erzeugten, ein Element der Menge darstellenden Folge von Zeichenreihen kan übrigens nach jeder Wahl beliebig (z.B. bis zur völligen Bestimmtheit, oder auch einem Mengengesetze entsprechend) verengert werden, und zwar stellt die Beliebigkeit dieser den einzelnen Wahlen unter Erhaltung der Fortsetzbarkeitsmöglichkeit zuzuordnenden Verengerungszusätze einen wesentlichen Charakter des freien Werdens des Mengenelementes dar. Jedem einzelnen Verengerungszusatz kann wieder ein die Beliebigkeit der weiteren Verengerungszusätze einschränkender Verengerungszusatz zweiter Ordnung beigegeben werden, usw.

3. Die Definition der individualisierten Menge⁹⁾ soll selbstverständlich erst nach der Definition der Verschiedenheit von Mengenelementen ihren Platz erhalten.

4. Die Definition der Halbidentität¹⁰⁾, in welche sich a.a.O. ein sinnstörender Druckfehler eingeschlichen hat, soll so gelesen werden, dass eine mit der Spezies N kongruente Teilspezies M von N mit N halbidentisch genannt wird.

⁸⁾ Mathem. Annalen, Bd. 93, S. 245, Fussnote³⁾.

⁹⁾ Mathem. Annalen, Bd. 93, S. 245, Z. 13 v.o.

¹⁰⁾ Mathem. Annalen, Bd. 93, S. 246, Z. 8 v.u.

Mathematics. — Ueber eine Formel aus der Komplexgeometrie. Von R. WETTZENBÖCK.

(Communicated at the meeting of March 28, 1942.)

Ich leite hier eine Formel ab, die zum Ausdruck bringt, dass die n Nullpunkte, die in einem Gebiete G_m zu m linearen G_{m-1} -Komplexen gehören, linear abhängig sind.

§ 1.

Im projektiven $G_n = R_{n-1}$ ordnet ein linearer G_{m-1} -Komplex

$$K_1 = (a^{n-m+1} \pi^{m-1}) = 0$$

einem Gebiete G_m mit den Koordinaten $\varphi_{i_1 i_2 \dots i_m}$ einem Nullpunkt P_1 zu mit der Gleichung

$$(P_1 u') = (a^{n-m} \varphi^m) (au') = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Geometrisch ist P_1 so definiert: alle G_{m-1} des Komplexes K_1 , die im $G_m(\varphi^m)$ liegen, gehen durch P_1 .

Es seien nun

$$K_1(a^{n-m+1}), K_2(b^{n-m+1}), \dots, K_{m-1}(g^{n-m+1}), K_m(h^{n-m+1})$$

$m \geq 2$ lineare G_{m-1} -Komplexe allgemeiner Lage. Im Gebiete $G_m(\varphi^m)$ liegen dann die m Nullpunkte

$$P_1, P_2, \dots, P_{m-1}, P_m.$$

Es entsteht dann die Frage: für welche Gebiete $G_m(\varphi^m)$ werden diese m Punkte P_i linear-abhängig, liegen also in einem G_{m-1} ?

Hierzu ist notwendig und hinreichend

$$(P_1 P_2 \dots P_m \pi^{n-m}) \equiv 0 \quad \{ \text{für alle } \pi_{i_1 \dots i_{n-m}} \}. \quad . \quad . \quad . \quad (2)$$

Setzt man hier die P_i nach (1) ein, so ergibt sich

$$F = (ab \dots gh \pi^{n-m}) (a^{n-m} \varphi_1^m) \dots (h^{n-m} \varphi_m^m) \equiv 0 \{ \pi \},$$

wobei $\varphi_1, \varphi_2, \dots, \varphi_m$ äquivalent sind, sodass also jeder Faktor $(\varphi_i^m \varphi_k \dots)$ zu Null führt.

Bringen wir nun in F die $n-m$ Reihen h des letzten Faktors in den ersten, so entsteht, bis auf einen konstanten Faktor $\neq 0$:

$$(ab \dots gh^{n-m+1}) (a^{n-m} \varphi_1^m) \dots (g^{n-m} \varphi_{m-1}^m) \cdot (\pi^{n-m} \varphi_m^m) \equiv 0 \{ \pi \},$$

d.h. wir erhalten an Stelle von (2):

$$D_{n,m}^* = (ab \dots gh^{n-m+1}) (a^{n-m} \varphi_1^m) \dots (g^{n-m} \varphi_{m-1}^m) = 0. \quad . \quad . \quad (3)$$

Die Gebiete $G_m(\varphi^m)$, für welche die m Nullpunkte linear-abhängig werden, gehören somit einem G_m -Komplexe vom Grade $m-1$ an.

Nach (1) kann man die Invariante (3) auch so schreiben:

$$D_{n,m}^* = (-1)^{m-1} (h^{n-m+1} P_1 P_2 \dots P_{m-1}). \quad (4)$$

Da überdies wegen

$$D_{n,m}^* = - (ab \dots hg^{n-m+1}) (a^{n-m} \varphi_1^m) \dots (h^{n-m} \varphi_{m-1}^m)$$

$D_{n,m}^*$ schief-symmetrisch in allen m Komplexen K ist, kann man nach (4) die Bedingung (3) auch so lesen: Das Gebiet G_{m-1} , das in $G_m (\varphi^m)$ von $m-1$ Nullpunkten bestimmt wird, muss dem m -ten Komplex angehören.

§ 2.

Wenn Q_1, Q_2, \dots, Q_m m linear-unabhängige Punkte des Gebietes $G_m (\varphi^m)$ sind, so sind

$$\lambda_1^1 Q_1 + \lambda_1^2 Q_2 + \dots + \lambda_1^m Q_m, \dots, \lambda_{m-1}^1 Q_1 + \lambda_{m-1}^2 Q_2 + \dots + \lambda_{m-1}^m Q_m$$

$m-1$ willkürliche Punkte des G_m , die man zu einem G_{m-1} verbinden kann. Soll dieses G_m dem Komplex $K_1 (a^{n-m+1})$ angehören, so ergibt sich

$$(a^{n-m+1}, \sum \lambda_1^i Q_i, \dots, \sum \lambda_{m-1}^i Q_i) = 0$$

oder, in leicht verständlicher Darstellung:

$$\sum_{j=1}^m (\lambda_1 \dots \lambda_{m-1})^{12 \dots j-1, j+1 \dots m} (a^{n-m+1} Q_1 Q_2 \dots Q_{j-1} Q_{j+1} \dots Q_m) = 0.$$

Hier sind die $(m-1)$ -reihigen Determinanten der λ_i^k die Koordinaten des G_{m-1} im G_m , wobei in diesem letzteren $Q_1 Q_2 \dots Q_m$ das Koordinatensimplex ist, sodass die m Determinanten

$$A_j = (a^{n-m+1} Q_1 \dots Q_{j-1} Q_{j+1} \dots Q_m) \quad (j=1, 2, \dots, m) \quad (5)$$

die Koordinaten des Nullpunktes P_1 bezüglich dieses Simplexes darstellen.

Die Bedingung für die Abhängigkeit der m Punkte P_i lässt sich jetzt so schreiben:

$$D_{n,m} = \begin{vmatrix} (a^{n-m+1} Q_2 Q_3 \dots Q_m) & (a^{n-m+1} Q_1 Q_3 \dots Q_m) & \dots & (a^{n-m+1} Q_1 Q_2 \dots Q_{m-1}) \\ (b^{n-m+1} Q_2 Q_3 \dots Q_m) & (b^{n-m+1} Q_1 Q_3 \dots Q_m) & \dots & (b^{n-m+1} Q_1 Q_2 \dots Q_{m-1}) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (h^{n-m+1} Q_2 Q_3 \dots Q_m) & (h^{n-m+1} Q_1 Q_3 \dots Q_m) & \dots & (h^{n-m+1} Q_1 Q_2 \dots Q_{m-1}) \end{vmatrix} = 0 \quad (6)$$

In diesem Ausdrucke müssen sich die Punkte Q_i zu den Koordinaten

$$\varphi_{i_1 \dots i_m} = (Q_1 Q_2 \dots Q_m)_{i_1 \dots i_m}$$

zusammenfassen lassen, sodass $D_{n,m}$ bis auf einem Zahlenfaktor mit $D_{n,m}^*$ von (3) identisch wird. Man erhält durch Entwicklung der Determinante (6) und Umformung die Beziehung

$$D_{n,m} = (-1)^{\frac{1}{2} m(m-1)} \left(\frac{n-m+1}{m!} \right)^{m-1} \cdot D_{n,m}^* \quad (7)$$

(Communicated at the meeting of March 28, 1942.)

CHAPTER IV.

*Identity theorems*¹⁾.

Let us consider the functional system

$$g_{\varrho} \{x_{\tau}, f_{\nu}(l_{\nu\omega}(x_{\tau}))\} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

consisting of n functional equations; κ assumes the values $1, 2, \dots, k$ (where $k \geq 2$), μ the values $1, 2, \dots, k-1$ (where $k-1 \geq 1$), ν and ϱ the values $1, 2, \dots, n$ (where $n \geq 1$) and finally τ and ω the values $1, 2, \dots, t$ (where $t \geq 1$); $g_{\varrho}(x_{\tau}, y_{\nu\tau})$ denote n given functions of the $t + kn$ variables $x_{\tau}, y_{\nu\tau}$, further $l_{\nu\omega}(x_{\tau})$ kt given functions of the t variables x_{τ} and ultimately $f_{\nu}(l_{\nu})$ n unknown functions of the t variables l_{ν} . The meaning is that (x_{τ}) denotes an arbitrary point of a given point aggregate \mathfrak{X} lying in the t -dimensional espace such that the origin is a limiting point of \mathfrak{X} which belongs to \mathfrak{X} . I assume that to any point (x_{τ}) of \mathfrak{X} correspond kt given numbers $l_{\nu\omega}(x_{\tau})$ taking the value zero for $x_1 = \dots = x_t = 0$. Let \mathfrak{L} be a point aggregate in the t -dimensional espace that contains the k points $\{l_{\nu 1}(x_{\tau}), \dots, l_{\nu t}(x_{\tau})\}$ for any point (x_{τ}) of \mathfrak{X} .

Now let us suppose that on \mathfrak{L} $2n$ functions $f_{\nu}(l_{\nu})$ and $f_{\nu}^*(l_{\nu})$ are given such that the functional system (1) and the system

$$g_{\varrho} \{x_{\tau}, f_{\nu}^*(l_{\nu\omega}(x_{\tau}))\} = 0$$

hold at any point (x_{τ}) of \mathfrak{X} ; then I say for the sake of brevity that the functional system (1) holds on \mathfrak{X} with $f_{\nu} = f_{\nu}$ and with f_{ν}^* for f_{ν} . In this chapter I shall establish sufficient conditions that the two solutions are identical at any point (l_{ν}) of \mathfrak{L} in the vicinity of the origin, more precisely, that there exists a vicinity (depending on the two considered solutions) of the origin with the property $f_{\nu}(l_{\nu}) = f_{\nu}^*(l_{\nu})$ at every point (l_{ν}) of \mathfrak{L} in this vicinity.

To begin with I state the conditions common to the first and second identity theorem. First I establish the condition concerning the sets \mathfrak{X} and \mathfrak{L} .

(1) *The origin $x_{\tau} = 0$ is a limiting point of \mathfrak{X} , that belongs to \mathfrak{X} and \mathfrak{L} contains for any point (x_{τ}) of \mathfrak{X} the k points $\{l_{\nu 1}(x_{\tau}), \dots, l_{\nu t}(x_{\tau})\}$. Conversely, to any point (l_{ν}) of \mathfrak{L} corresponds one and only one point (x_{τ}) of \mathfrak{X} , satisfying the t relations $l_{k\omega}(x_{\tau}) = l_{\nu}$. If the point (x_{τ}) of \mathfrak{X} approaches the origin, then so does the point $\{l_{k1}(x_{\tau}), \dots, l_{kt}(x_{\tau})\}$ of \mathfrak{L} and conversely.*

Next the condition imposed on the functions $g_{\varrho}(x_{\tau}, y_{\nu\tau})$.

(2) *For any point (x_{τ}) of \mathfrak{X} and any point $(y_{\nu\tau})$ in the vicinity of the origin $y_{\nu\tau} = 0$*

¹⁾ Chapter I and the first part of chapter II have been published in *Euclides* 18 (1941—42), p. 50—78, and chapter III in *Proc. Ned. Akad. v. Wetensch.*, Amsterdam, 45, 129—135, 217—224 (1942). The rest of chapter II is about to appear in *Euclides*. For the well understanding of chapter IV it is not necessary that the reader is acquainted with the preceding chapters.

of the kn -dimensional space the kn^2 partial derivatives $\frac{\partial g_\varrho}{\partial y_{\kappa\nu}}$ exist and tend to $\left(\frac{\partial g_\varrho}{\partial y_{\kappa\nu}}\right)_0^1$, as the point (x_τ) of \mathfrak{X} approaches the origin $x_\tau=0$ and the point $(y_{\kappa\nu})$ the origin $y_{\kappa\nu}=0$. The n^2 derivatives $\left(\frac{\partial g_\varrho}{\partial y_{\kappa\nu}}\right)_0$ form a determinant $\Delta \neq 0$ and $g_\varrho(x_\tau, y_{\kappa\nu})=0$ at $x_\tau=y_{\kappa\nu}=0$.

Further a condition, involving the kt functions $l_{\kappa\omega}(x_\tau)$ and the kn linear forms

$$L_{\kappa\nu}(w_\varrho) = \sum_{\varrho} \left(\frac{\partial g_\nu}{\partial y_{\kappa\nu}} \right) w_\varrho.$$

(3) The values which the positive-homogeneous function $H(z_\varrho)$ of the first degree²⁾ of the n variables z_ϱ assumes on the hypersphere $\sum |z_\varrho|^2 = 1$ lie between two positive finite bounds. To every point (l_ω) of \mathfrak{L} corresponds a number $G(l_\omega) \geq 0$, which is positive for $(l_\omega) \neq 0^3)$, such that $G\{l_{\mu\omega}(x_\tau)\} : G\{l_{k\omega}(x_\tau)\}$ is bounded for every point $(x_\tau) \neq 0$ of \mathfrak{X} and $\mu = 1, \dots, k-1$, and that

$$\sum_{\mu} G\{l_{\mu\omega}(x_\tau)\} H\{L_{\mu\nu}(w_\varrho)\} \leq \Theta G\{l_{k\omega}(x_\tau)\} H\{L_{k\nu}(w_\varrho)\} \quad . \quad . \quad (2)$$

for every point (x_τ) of \mathfrak{X} and any system (w_ϱ) , where Θ denotes a positive constant ≤ 1 .

In the special case in which the considered functional system consists of only one functional equation involving one unknown function, we have $n=1$ and we can take $H(z_\varrho) = |z_1|$, so that (2) reduces to

$$\sum_{\mu} G\{l_{\mu\omega}(x_\tau)\} \left| \frac{\partial g_1}{\partial y_{\mu 1}} \right|_0 \leq \Theta G\{l_{k\omega}(x_\tau)\} \left| \frac{\partial g_1}{\partial y_{k 1}} \right|_0 \quad . \quad . \quad . \quad (3)$$

Finally a condition involving an auxiliary function $J(l_\omega)$.

(4) To every point (l_ω) of \mathfrak{L} corresponds a number $J(l_\omega) \geq 0$ such that $J(l_\omega)$ tends to zero, as (l_ω) approaches the origin and conversely, and that for every point (x_τ) of \mathfrak{X}

$$J\{l_{\mu\omega}(x_\tau)\} \leq \vartheta J\{l_{k\omega}(x_\tau)\} \quad (\mu = 1, \dots, k-1), \quad . \quad . \quad . \quad (4)$$

where ϑ is a positive number ≤ 1 , independent of μ and (x_τ) .

First identity theorem.

Suppose that the conditions (1), (2), (3) and (4) are satisfied with $\Theta < 1$ and that the considered functional system holds on \mathfrak{X} with $f_\nu = f_\nu$ and with f_ν^* for f_ν . Let us further assume that $f_\nu(l_\omega) = f_\nu^*(l_\omega) = 0$ at the origin $l_\omega = 0$ and that

$$f_\nu(l_\omega) \rightarrow 0, \quad f_\nu^*(l_\omega) \rightarrow 0, \quad \frac{f_\nu^*(l_\omega) - f_\nu(l_\omega)}{G(l_\omega)} \text{ is bounded}, \quad . \quad . \quad . \quad (5)$$

as the point $(l_\omega) \neq 0$ of \mathfrak{L} approaches the origin.

Then the two solutions $(f_\nu(l_\omega))$ and $(f_\nu^*(l_\omega))$ are identical at the points (l_ω) of \mathfrak{L} in the vicinity of the origin.

¹⁾ The suffix signifies that the value is meant which the derivative takes at the origin $x_\tau = y_{\kappa\nu} = 0$.

²⁾ In other words: $H(uz_\varrho) = uH(z_\varrho)$ for any $u \geq 0$.

³⁾ The notation $(l_\omega) \neq 0$ signifies that (l_ω) does not coincide with the origin $l_\omega = 0$.

Second identity theorem.

Suppose that the conditions (1), (2), (3) and (4) are satisfied with $\vartheta < 1$ and that the kn^2 relations

$$\frac{\partial g_e}{\partial y_{zv}} = \left(\frac{\partial g_e}{\partial y_{zv}} \right)_0 + O \{ J^q(l_{k\omega}(x_\tau)) + \sum_{\substack{\lambda=1, \dots, k \\ \sigma=1, \dots, n}} |y_{\lambda\sigma}|^q \} \quad (6)$$

are true for any point (x_τ) of \mathfrak{X} in the vicinity of the origin $x_\tau = 0$ and any point (y_{zv}) in the vicinity of the origin $y_{zv} = 0$, where q denotes a positive constant.

Let us further assume that the considered functional system holds on \mathfrak{X} with $f_v = f_v$ and with f_v^* for f_v with the property that

$$\frac{f_v(l_n)}{J^q(l_n)} \quad \text{and} \quad \frac{f_v^*(l_n)}{J^q(l_n)} \quad \text{are bounded and} \quad \frac{f_v^*(l_n) - f_v(l_n)}{G(l_n)} \rightarrow 0, \quad (7)$$

as the point $(l_n) \neq 0$ of \mathfrak{L} approaches the origin $l_n = 0$.

Then the two solutions are identical at the points (l_n) of \mathfrak{L} in the vicinity of the origin.

If $\Theta < 1$, it is of course recommendable to apply the first identity theorem.

To prove both theorems, I begin with the condition (3). Let \mathfrak{B} be the set of the points (z_v) satisfying for any system (w_v) the inequality

$$|\Re \sum_v w_v z_v| \leq H(w_v). \quad (8)$$

This aggregate is convex, for if it contains two points (z_v) and (z'_v) , it contains also (z''_v) , where $z''_v = \lambda z_v + (1-\lambda) z'_v$ ($0 \leq \lambda \leq 1$), since $\Re \sum_v w_v z''_v$ is on the one hand

$$\leq \lambda H(w_v) + (1-\lambda) H(w_v) = H(w_v)$$

and on the other hand $\geq -H(w_v)$.

\mathfrak{B} is closed, for if a point (z_v) belonging to it tends to (z_v^*) , the inequality (8) holds with z_v^* for z_v . If (z_v) belongs to \mathfrak{B} , then so does $(-z_v)$, so that the origin is a centre of \mathfrak{B} . If η_1 and η_2 denote the lower and upper bounds of $H(w_e)$ on the hypersphere $\sum_e |w_e|^2 = 1$, we have for any point $(w_v) \neq 0$

$$H^2(w_e) = \left(\sum_v |w_v|^2 \right) H^2 \left(\frac{w_e}{\sqrt{\sum_v |w_v|^2}} \right) \geq \eta_1^2 \sum_v |w_v|^2$$

and it follows from

$$|\Re \sum_v z_v w_v|^2 \leq \left(\sum_v |w_v|^2 \right) \left(\sum_v |z_v|^2 \right)$$

that the points (z_v) with $\sum_v |z_v|^2 \leq \eta_1^2$ belong to \mathfrak{B} ; hence the origin is an inner point of \mathfrak{B} .

Denoting by w_v the complex number conjugate to z_v , we find for any point (z_v) with $\sum_v |z_v|^2 > \eta_2^2$

$$\Re \sum_v w_v z_v = \sqrt{\left(\sum_v |w_v|^2 \right) \left(\sum_v |z_v|^2 \right)} > \eta_2 \sqrt{\sum_v |w_v|^2} \geq H(w_v),$$

so that these points (z_v) do not belong to \mathfrak{B} ; hence \mathfrak{B} is bounded.

We may interpret (2) as follows: if \mathfrak{B} contains $k-1$ points $(z_{\mu 1}, \dots, z_{\mu n})$ and (x_τ) denotes an arbitrary point $\neq 0$ of \mathfrak{X} , then \mathfrak{B} contains also the point (z_ν) , defined by the n relations

$$\Theta G \{l_{k\alpha}(x_\tau)\} \sum_\nu \left(\frac{\partial g_\varrho}{\partial y_{k\nu}} \right)_0 z_\nu = - \sum_{\mu, \nu} G \{l_{\mu\alpha}(x_\tau)\} \left(\frac{\partial g_\varrho}{\partial y_{\mu\nu}} \right)_0 z_{\mu\nu}; \quad (9)$$

this point (z_ν) is defined unambiguously in virtue of

$$\Theta > 0, \quad G \{l_{k\alpha}(x_\tau)\} > 0 \quad \text{and} \quad \Delta \neq 0. \quad (10)$$

To prove this assertion, it is sufficient to deduce (8) for any system (w_ν) . Using (10) we find n numbers s_ϱ satisfying the n equations

$$w_\nu = \Theta G \{l_{k\alpha}(x_\tau)\} \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{k\nu}} \right)_0 s_\varrho \quad (11)$$

and we obtain

$$\begin{aligned} \Re \sum_\nu w_\nu z_\nu &= \Theta G \{l_{k\alpha}(x_\tau)\} \Re \sum_\varrho s_\varrho \sum_\nu \left(\frac{\partial g_\varrho}{\partial y_{k\nu}} \right)_0 z_\nu \\ &= - \sum_\mu G \{l_{\mu\alpha}(x_\tau)\} \Re \sum_\nu z_{\mu\nu} \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{\mu\nu}} \right)_0 s_\varrho \end{aligned}$$

by means of (9). Since the $k-1$ points $(z_{\mu 1}, \dots, z_{\mu n})$ belong to \mathfrak{B} , the formula (8), applied with $z_{\mu\nu}$ for z_ν and with $\sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{\mu\nu}} \right)_0 s_\varrho$ for w_ν , gives

$$\begin{aligned} \left| \Re \sum_\nu w_\nu z_\nu \right| &\leq \sum_\mu G \{l_{\mu\alpha}(x_\tau)\} H \left\{ \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{\mu\nu}} \right)_0 s_\varrho \right\} \\ &\leq \Theta G \{l_{k\alpha}(x_\tau)\} H \left\{ \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{k\nu}} \right)_0 s_\varrho \right\} \end{aligned}$$

by virtue of (2)

$$= H \left\{ \Theta G \{l_{k\alpha}(x_\tau)\} \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{k\nu}} \right)_0 s_\varrho \right\}$$

since $H(w_\nu)$ is a positive-homogeneous function of z_ν of the first degree. Therefore (11) gives (8) and (z_ν) belongs to \mathfrak{B} .

Proof of the first identity theorem.

Any point (x_τ) of \mathfrak{X} satisfies the n relations

$$g_\varrho(x_\tau, f_{\nu\nu}^*) - g_\varrho(x_\tau, f_{\nu\nu}) = 0, \quad (12)$$

where

$$f_{\nu\nu}^* = f_\nu^* \{l_{\alpha\alpha}(x_\tau)\} \quad \text{and} \quad f_{\nu\nu} = f_\nu \{l_{\alpha\alpha}(x_\tau)\},$$

the two terms on the left-hand side of (12) being equal to zero. In the neighbourhood

of the origin $\frac{\partial g_e}{\partial y_{xv}}$ is continuous and therefore approximatively equal to $\left(\frac{\partial g_e}{\partial y_{xv}}\right)_0$ (if (x_τ) belongs to \mathfrak{X}), hence

$$\sum_{x,v} \left(\frac{\partial g_e}{\partial y_{xv}} \right)_0 (f_{xv}^* - f_{xv}) + o \sum_{x,v} |f_{xv}^* - f_{xv}| = 0.$$

Putting

$$f_v^*(l_\omega) = f_v(l_\omega) + G(l_\omega) Q_v(l_\omega) \quad \text{at } (l_\omega) \neq 0 \quad . \quad . \quad . \quad (13)$$

and $Q_v(l_\omega) = 0$ at the origin $l_\omega = 0$, the functions $Q_v(l_\omega)$ are bounded on \mathfrak{L} in the vicinity of the origin. If the point $(x_\tau) \neq 0$ of \mathfrak{X} lies in the vicinity of the origin, we find so, since $G\{l_{\mu\omega}(x_\tau)\} : G\{l_{k\omega}(x_\tau)\}$ is bounded,

$$\begin{aligned} \sum_{x,v} \left(\frac{\partial g_e}{\partial y_{xv}} \right)_0 G\{l_{\nu\omega}(x_\tau)\} Q_v\{l_{x\omega}(x_\tau)\} &= o \sum_{x,v} G\{l_{x\omega}(x_\tau)\} |Q_v\{l_{x\omega}(x_\tau)\}| \\ &= o G\{l_{k\omega}(x_\tau)\} S, \end{aligned}$$

where

$$S = \sum_{x,v} |Q_v\{l_{x\omega}(x_\tau)\}|.$$

Putting $Q_v\{l_{\mu\omega}(x_\tau)\} = P_{\mu v}$ and defining P_v by the n equations

$$G\{l_{k\omega}(x_\tau)\} \sum_v \left(\frac{\partial g_e}{\partial y_{kv}} \right)_0 P_v = - \sum_{\mu,v} G\{l_{\mu\omega}(x_\tau)\} \left(\frac{\partial g_e}{\partial y_{\mu v}} \right)_0 P_{\mu v}, \quad . \quad (14)$$

we obtain by taking the difference and dividing by $G\{l_{k\omega}(x_\tau)\}$

$$\sum_v \left(\frac{\partial g_e}{\partial y_{kv}} \right)_0 \{Q_v(l_{k\omega}(x_\tau)) - P_v\} = o(S),$$

hence, on account of $\Delta \neq 0$,

$$Q_v\{l_{k\omega}(x_\tau)\} - P_v = o(S). \quad . \quad . \quad . \quad (15)$$

The object of this proof is to show that $Q_v(l_\omega) = 0$ at the points (l_ω) of \mathfrak{L} in the vicinity of the origin. Suppose there exists in every vicinity of the origin $l_\omega = 0$ a point (l_ω) of \mathfrak{L} with $\sum_v |Q_v(l_\omega)| > 0$. Since the origin is an inner point of \mathfrak{B} , this aggregate contains the points with the n coordinates $\xi Q_v(l_\omega)$, if the positive number ξ is small enough. \mathfrak{B} being bounded, the set of these numbers ξ possesses an upper bound $\psi(l_\omega)$. Since \mathfrak{B} is closed, it contains the point with the n coordinates $\psi(l_\omega) Q_v(l_\omega)$.

The point (x_τ) of \mathfrak{X} being defined by the n equations $l_{k\omega}(x_\tau) = l_\omega$, I shall show: if the point (l_ω) of \mathfrak{L} with $\sum_v |Q_v(l_\omega)| > 0$ lies near enough to the origin, then it is excluded that \mathfrak{B} contains the $k-1$ points with the coordinates

$$\frac{1}{2}(1 + \Theta) \psi(l_\omega) P_{\mu v} = \frac{1}{2}(1 + \Theta) \psi(l_\omega) Q_v\{l_{\mu\omega}(x_\tau)\}. \quad . \quad . \quad . \quad (16)$$

In fact, suppose that every vicinity of the origin $l_\omega = 0$ contains a point (l_ω) of \mathfrak{L} with $\sum_v |Q_v(l_\omega)| > 0$ and with the property that the said $k-1$ points belong to \mathfrak{B} . Then the coordinates of these points and also the coordinates $\psi(l_\omega) Q_v\{l_{k\omega}(x_\tau)\}$ are bounded, so that $\psi(l_\omega) S$ is bounded and (15) reduces to

$$\{Q_v(l_\omega) - P_v\} \psi(l_\omega) = o(1). \quad . \quad . \quad . \quad (17)$$

The origin being an inner point of \mathfrak{B} , this set contains therefore the point with the n coordinates

$$\frac{(1+\Theta)(1+3\Theta)}{2\Theta(1-\Theta)} \psi(l_o) \{Q_v(l_o) - P_v\},$$

if (l_o) lies near enough to the origin.

\mathfrak{B} containing the $k-1$ points with the coordinates (16), it follows from the property found above for this set, that it contains also the point with the n coordinates $\frac{1}{2\Theta}(1+\Theta)\psi(l_o)P_v$ where P_v is defined by (14). On account of

$$\frac{1+3\Theta}{2+2\Theta} + \frac{1-\Theta}{2+2\Theta} = 1$$

the convex set \mathfrak{B} contains therefore also the point with the n coordinates

$$\begin{aligned} \frac{1+3\Theta}{2+2\Theta} \frac{1+\Theta}{2\Theta} \psi(l_o)P_v + \frac{1-\Theta}{2+2\Theta} \frac{(1+\Theta)(1+3\Theta)}{2\Theta(1-\Theta)} \psi(l_o) \{Q_v(l_o) - P_v\} = \\ = \zeta Q_v(l_o), \end{aligned}$$

where

$$\zeta = \frac{1+3\Theta}{4\Theta} \psi(l_o) > \psi(l_o).$$

Hence, we have a contradiction to the assumption that $\psi(l_o)$ is the upper bound of the numbers ζ with the property that \mathfrak{B} contains the point with the n coordinates $\zeta Q_v(l_o)$. In this manner we have proved that there is a vicinity \mathfrak{B} of the origin with the following property: if \mathfrak{B} contains a point (l_o) of \mathfrak{L} with $\sum_v |Q_v(l_o)| > 0$, then it is excluded that

\mathfrak{B} contains the $k-1$ points with the coordinates (16).

The point (l_o) approaches the origin, as $J(l_o)$ tends to zero; hence (l_o) belongs to \mathfrak{B} , if $J(l_o)$ is small enough, say $\leq p$. It is sufficient to prove that $Q_v(l_o) = 0$ for all points (l_o) of \mathfrak{L} with $J(l_o) \leq p$; in fact, the points (l_o) in the vicinity of the origin satisfy the inequality $J(l_o) \leq p$.

Suppose there exists a point (l_o) of \mathfrak{L} with $J(l_o) \leq p$ and $\sum_v |Q_v(l_o)| > 0$. This point lying in \mathfrak{B} , there is, according to the above result, one μ at least ($1 \leq \mu \leq k-1$), such that \mathfrak{B} does not contain the point with the n coordinates (16), where $l_{k_o}(x_\tau) = l_o$. For this μ (if I have the choice, I take the smallest value) I put $l_{\mu o}(x_\tau) = l'_o$, so that (l'_o) is a point of \mathfrak{L} by the condition (1) and the point with the n coordinates $\frac{1}{2}(1+\Theta)\psi(l_o)Q_v(l'_o)$ does not belong to \mathfrak{B} . Hence

$$\sum_v |Q_v(l'_o)| > 0 \quad \text{and} \quad \psi(l'_o) \leq \frac{1+\Theta}{2} \psi(l_o).$$

Applying the condition (4) we obtain

$$J(l'_o) \leq J(l_o) \leq p.$$

Now we can repeat the argument with l'_o for l_o , and so we find a point (l''_o) of \mathfrak{L} with

$$\sum_v |Q_v(l''_o)| > 0; \quad \psi(l''_o) \leq \frac{1+\Theta}{2} \psi(l'_o) \leq \left(\frac{1+\Theta}{2}\right)^2 \psi(l_o) \quad \text{and} \quad J(l''_o) \leq p,$$

and so on. In this manner we find an infinity of points (l_{ω}^*) , all belonging to \mathfrak{Q} , such that

$$\sum_{\nu} |Q_{\nu}(l_{\omega}^*)| > 0; \quad \psi(l_{\omega}^*) \rightarrow 0 \quad \text{and} \quad J(l_{\omega}^*) \leq p.$$

The last inequality shows that these points (l_{ω}^*) belong to the vicinity \mathfrak{B} of the origin, the last but one that $\sum_{\omega} |l_{\omega}^*|$ increases indefinitely. Thus we have found a contradiction and the theorem is established.

Proof of the second identity theorem.

The proof is similar to the previous one. We may suppose without loss of generality that $q \leq 1$. The formula (12) holds again and the left-hand side is by virtue of (6) equal to

$$\sum_{x, \nu} \left(\frac{\partial g_{\nu}}{\partial y_{\nu}} \right)_0 (f_{x\nu}^* - f_{x\nu}) + O \{ J^q(l_{k\omega}(x_{\tau})) + \sum_{x, \nu} |f_{x\nu}|^q + \sum_{x, \nu} |f_{x\nu}^*|^q \} \sum_{x, \nu} |f_{x\nu}^* - f_{x\nu}|.$$

The last term is at most of the same order as

$$J^m \{ l_{k\omega}(x_{\tau}) \} \sum_{x, \nu} |f_{x\nu}^* - f_{x\nu}|,$$

where $m = q^2$, as it follows from $q \leq 1$, (7) and (4). Defining $Q_{\nu}(l_{\omega})$, S , $P_{\mu\nu}$ and P_{ν} as in the previous proof, I find in place of (15)

$$|Q_{\nu} \{ l_{k\omega}(x_{\tau}) \} - P_{\nu}| \leq K S J^m \{ l_{k\omega}(x_{\tau}) \}, \quad . \quad . \quad . \quad (18)$$

K denoting a convenient constant. If M and N denote appropriate positive numbers, \mathfrak{B} contains every point, the coordinates of which are all in absolute value $\leq M$ and \mathfrak{B} does not contain any point of which one or more coordinates are in absolute value $> N$.

Putting $T = \frac{2knNK}{M}$ and defining $\psi(l_{\omega})$ as in the previous proof, I assert: if a point (l_{ω}) of \mathfrak{Q} with $\sum_{\nu} |Q_{\nu}(l_{\omega})| > 0$ lies near enough to the origin, it is excluded that \mathfrak{B} contains the $k-1$ points with the coordinates

$$\{ 1 + T J^m(l_{\omega}) \} \psi(l_{\omega}) P_{\mu\nu} = \{ 1 + T J^m(l_{\omega}) \} \psi(l_{\omega}) Q_{\nu} \{ l_{\mu\omega}(x_{\tau}) \}, \quad . \quad (19)$$

where $l_{k\omega}(x_{\tau}) = l_{\omega}$. In fact, suppose that every vicinity of the origin $l_{\omega} = 0$ contains a point (l_{ω}) of \mathfrak{Q} with $\sum_{\nu} |Q_{\nu}(l_{\omega})| > 0$ and with the property that \mathfrak{B} contains the $k-1$ points with the coordinates (19). Then these coordinates and also the coordinates $\psi(l_{\omega}) Q_{\nu}(l_{k\omega}(x_{\tau}))$ are in absolute value $\leq N$, so that $|\psi(l_{\omega}) S| \leq knN$ and (18) reduces to

$$|Q_{\nu}(l_{\omega}) - P_{\nu}| \psi(l_{\omega}) \leq knNK J^m(l_{\omega}).$$

The numbers

$$\frac{Q_{\nu}(l_{\omega}) - P_{\nu}}{\frac{1}{2} T J^m(l_{\omega})} \psi(l_{\omega})$$

are therefore in absolute value $\leq \frac{knNK}{\frac{1}{2}T} = M$, so that the point with these coordinates belongs to \mathfrak{B} . This set, containing the $k-1$ points with the coordinates (19), contains also the point with the coordinates $\{1 + TJ^m(l_o)\} \psi(l_o) P_v$, where P_v is defined by (14), so that it contains moreover the point with the coordinates

$$(1-\lambda) \{1 + TJ^m(l_o)\} \psi(l_o) P_v + \lambda \frac{Q_v(l_o) - P_v}{\frac{1}{2}TJ^m(l_o)} \psi(l_o) = \zeta Q_v(l_o),$$

where

$$\begin{aligned} \zeta &= (1-\lambda) \{1 + TJ^m(l_o)\} \psi(l_o) = \frac{\lambda}{\frac{1}{2}TJ^m(l_o)} \psi(l_o) \\ &= \frac{1 + TJ^m(l_o)}{1 + \frac{1}{2}TJ^m(l_o) + \frac{1}{2}T^2J^{2m}(l_o)} \psi(l_o) > \psi(l_o), \end{aligned}$$

if $J(l_o)$ is small enough, that is, if (l_o) lies near enough to the origin.

In this manner we obtain a contradiction and we find that there exists a positive number p such that to any point (l_o) of \mathfrak{L} with $\sum_v |Q_v(l_o)| > 0$ and $J(l_o) \leq p$ corresponds one μ at least ($1 \leq \mu \leq k-1$) with the property that the point with the n coordinates (19) does not belong to \mathfrak{B} . If more than one such value of μ is possible, I choose the smallest value with this property; I put $l_{\mu o}(x_\tau) = l'_{\mu o}$. The point with the n coordinates $(1 + TJ^m(l_o)) \psi(l_o) Q_v(l'_{\mu o})$ does not belong to \mathfrak{B} , so that

$$\sum_v |Q_v(l'_{\mu o})| > 0 \quad \text{and} \quad \psi(l'_{\mu o}) \leq \{1 + TJ^m(l_o)\} \psi(l_o);$$

from the condition (4) it follows that

$$J(l'_{\mu o}) \leq \vartheta J(l_o) \leq \vartheta p$$

In the same manner we find a point $(l''_{\mu o})$ of \mathfrak{L} satisfying the inequalities

$$\psi(l''_{\mu o}) \leq \{1 + TJ^m(l'_{\mu o})\} \psi(l'_{\mu o}) \quad \text{and} \quad J(l''_{\mu o}) \leq \vartheta^2 p,$$

and so on. Thus we obtain an infinity of points (l_o^*) , all belonging to \mathfrak{L} , such that

$$J(l_o^*) \rightarrow 0 \quad \text{and} \quad \psi(l_o^{(h+1)}) \leq (1 + T\vartheta^{mh} p^m) \psi(l_o^{(h)}),$$

and therefore

$$\psi(l_o^*) \leq \psi(l_o) \prod_{h=0}^{\infty} (1 + T\vartheta^{mh} p^m) \quad . \quad . \quad . \quad . \quad (20)$$

$J(l_o^*) \rightarrow 0$ implies that (l_o^*) approaches the origin $l_o = 0$, so that it follows from (7) that the point with the n coordinates $Q_v(l_o^*)$ approaches the origin $z_v = 0$. Hence $\psi(l_o^*)$ increases indefinitely, contradictory to (20). This proves the theorem.

(To be continued.)

(Communicated at the meeting of February 28, 1942.)

The measurement of the changes in *surface potential* was made using YAMINS and ZISMAN's method¹⁾. A gold plate, placed close above the surface of the liquid is made to vibrate. Plate and liquid together form a condensor which is connected with the grid circuit of a valve detector. The alternating current generated is amplified and is made audible with a telephone; when the potential difference between liquid and gold plate is compensated by means of a potentiometer, the sound in the telephone reaches a minimum. The gold plate was set vibrating with an electrically driven tuning-fork instead of with the loudspeaker vibrator used by the authors mentioned, as with the first method we got less difficulties in screening the vibration source. The vibration amplitude was made very small so as to prevent the surface of the liquid coming into vibration too much. To amplify the alternating current produced we used a three set amplifier. The first tube was a Philips E 446, a tube with a very high amplifying factor. The limits of error of the measurements were ± 1 mV., the duplicatibility, however, was much less.

The *viscosity* measurements were made with a torsion pendulum²⁾ ³⁾ which method is the most suitable for films with rather high viscosities. As rotating body a massive gilt brass cylinder (27 mm diam.) was used, as torsion wire a very thin phosphorbronze wire (50 cm length). With a lense and a mirror fixed on top of the cylinder just beneath the point of attachment of the wire, the image of an illuminated arrow was formed on a circular scale (1 m diam.) The moment of inertia of the system was 230 g cm², the torsion constant of the wire 86 g cm² sec⁻², the period of oscillation 10.4 sec. At viscosities lower than about 8 g sec⁻¹ the logarithmic decrement of the oscillations was measured; above viscosities of about 200 g sec⁻¹ the motion was aperiodic so that the decrease of the amplitude with time could be measured; in the region between these viscosities no measurements were made. With a paraffined glass rod placed across the spreading through a square with an edge of 14 cm was obtained, the glass rod, the two long edges of the trough and the piston of the differential balance being the border.

Surface potentials for gliadin and gliadin-tannic acid films.

In fig. 4 the difference in surface potential for buffers with and without films upon it (ΔV) is plotted against the surface concentration at different pH values. In doing so one gets an idea of the concentration and orientation of the molecules in the surface, for it may be expected that ΔV increases almost proportionately to the number of molecules present in the surface when the orientation does not change and the molecules only little influence each other.

Under a pressure of 0.5–1 dn/cm the potential was very inconstant, which points to the film being inhomogeneous. Above this pressure the variations in surface potential were within the limit of error of the measurements (± 1 mV), except in the region of 7 dn/cm and higher in some cases. In these cases, moreover, the minimum in the telephone was not sharp, which, according to YAMINS and ZISMAN¹⁾ points to the presence of inhomogenities in the film. Sometimes the potential was even 50 m.V. under the normal values over a considerable pressure range. Then at higher pressures the potential increased rather rapidly, so that in the region where ΔV only slightly changed

1) H. G. YAMINS and W. A. ZISMAN, J. chem. Physics, **1**, 656 (1933).

2) I. LANGMUIR, J. Am. chem. Soc. **59**, II, 2410 (1937).

3) M. JOLY, J. chim. Phys. **36**, 285 (1939). Kolloid Z. **89**, 26 (1939).

with the concentration, the reproducibility was always good. These phenomena are presumably caused by the transition from the liquid to the gelatinous state already mentioned in our first article.

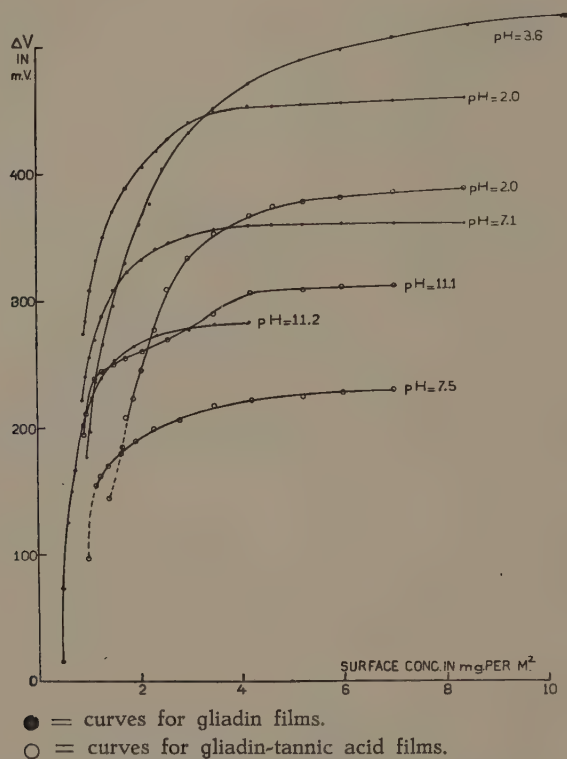


Fig. 4. Surface potential differences of gliadin and gliadin-tannic acid films, as a function of the surface concentration.

The curves show that at the lower concentrations ΔV increases rapidly and almost proportionately with the concentration, which indicates a practically constant „dipole moment”; in this region presumably free water molecules are squeezed out of the spaces in and between the micellae. At higher concentrations the increase of ΔV becomes smaller and smaller. COCKBAIN and SCHULMAN⁴⁾ have made it very comprehensible that in this region the orientation of the polypeptide chains changes. It is also possible, however, that hydration water (bound to the polar groups of the gliadin) is pressed out of the film as seems to be the case, according to LANGMUIR, with films of fatty acids; this would also cause a decrease of the ΔV increase (see PHILIPPI⁵⁾). At very high surface concentrations ΔV finally becomes practically constant, probably because the closest packing is obtained and gliadin molecules are pressed out of the film on further compression. In full agreement with this is the fact that the constant ΔV values always occur at a pressure of about 20 dn/cm, the same pressure at which the compressibility strongly increases and the pressure falls rapidly after each compression. In the case of gliadin-tannic acid films the constant ΔV values occur at pressures above 30 dn/cm (see also fig. 1 and 4). This latter observation is a strong indication that the decrease

⁴⁾ E. G. COCKBAIN and J. H. SCHULMAN, *Trans. Faraday Soc.* **35**, 1266 (1939).

⁵⁾ G. TH. PHILIPPI, *On the nature of proteins*. Thesis Leyden 1936.

in pressure after compression above 1 dn/cm has not to be considered as collapsing of the film, but that this occurs only at pressures above 30–40 dn/cm.

In fig. 5 the well reproducible final ΔV values are plotted against the pH. This figure shows that in the case of gliadin at low pH values ΔV is very great and passes a maximum; at higher pH values ΔV decreases rather rapidly. For the explanation of this phenomenon we refer to the detailed considerations of PHILIPPI⁵⁾. His work makes it very probable that the decrease of ΔV at increasing pH is not to be ascribed

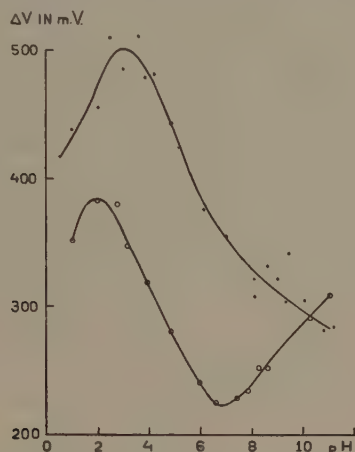


Fig. 5. Maximum surface potential differences of gliadin and gliadin-tannic acid films as a function of the pH.

● = curve for gliadin.

○ = curve for gliadin-tannic acid.

to a change in the orientation of the molecules, but is caused by changes in the ionization of the protein and by changes in the electric double layer.

At low pH values the ΔV -pH curves for gliadin-tannic acid films are of the same type as those for gliadin; only ΔV is on a lower level. At pH=7, however, ΔV increases rather rapidly. This is caused by the gradual transition from the gliadin-tannic acid complex into gliadin at pH values above 7. At pH=10–11 the influence of tannic acid has disappeared. This is in full agreement with the phenomena seen at film-pressure and compressibility measurements. The nearly constant ΔV difference (about 140 m.V.) between the complex and the gliadin films at pH values 1–7 indicates that in this region, presumably, a strongly ionised gliadin-tannic acid complex is present that is not influenced by the pH.

Viscosities of gliadin and gliadin-tannic acid films.

The viscosity of gliadin films is measured by determining the logarithmic decrement of torsion oscillations. The reproducibility was bad. When putting the torsion cylinder on the liquid surface and spreading after that, mostly a much lower viscosity was found than in case the cylinder was placed on the film already present; this must be caused by insufficient adherence of the film to the cylinder in the former case. We have, therefore, always applied the latter method. The lowering of the cylinder upon the surface had to be performed very carefully as small oscillations of the cylinder mostly gave very low viscosities (approximately those of pure water); this must be caused by disruption of the film at the place of adherence as a result of the vibration. The viscosity proved to be highly dependent on the amplitude of the oscillations; the smaller the amplitude the greater the viscosity was. This proves that the viscosity is strongly influenced by the

shearing stress. The viscosities plotted in the graphs are those measured with the smallest amplitude (oscillation over about one degree). A more accurate establishment of the viscosity was of no avail with these inaccurate measurements. Above pressures of about 10 dn/cm the gliadin films showed elasticity; the period of oscillation decreased, at pressures of 20 dn/cm even to 6 sec. instead of 10.4 sec.

On measuring the viscosity of one film consecutively at different pressures a rather smooth viscosity-pressure curve was obtained; sometimes big differences from 100 % more to 50 % less were noticed, however, when duplicating the measurement on another film. We were strongly under the impression that these differences were due to small variations in the history of the film (e.g. time between spreading and measuring, velocity of blowing out the pipette); only, the exact reason we do not know. JOLY³⁾ also noticed a strong influence of the history in the case of gliadin films; this author even distinguishes two types of films viz. highly viscous gel-like films, obtained by spreading in such a way that the pressure was high from the very beginning, and low viscous, non gel-like films obtained by spreading at a low initial pressure. These latter films remained, also at rather high pressures, relatively low viscous; the former films were more viscous according as the initial pressure was higher. In our method of spreading we could not obtain the low viscous type of films. The viscosities found in our measurements tally fairly well with those of the highly viscous films of JOLY.

The change in viscosity with pressure is shown in fig. 6 (curve I). All measurements were made by determining the damping of the oscillations. At a pressure of 6 dn/cm the

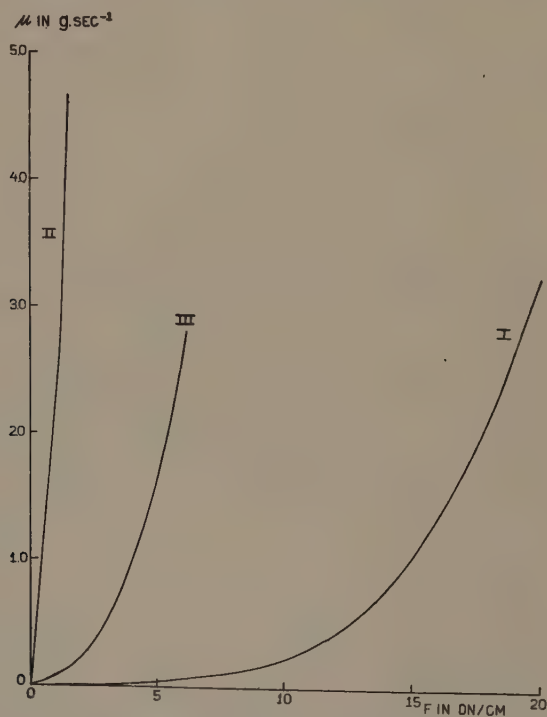


Fig. 6. Viscosity of gliadin films and of gliadin-tannic acid films at different surface pressures F .

- | | |
|-----|--|
| I | average curve for gliadin films at pH = 1—11 and for a gliadin-tannic acid film at pH = 10.7 |
| II | " " for gliadin-tannic acid film at pH = 1—6.5 |
| III | " " for gliadin-tannic acid film at pH = 7.5 |

viscosity of gliadin films is, on an average, twice that of a surface of pure water (about 0.03 g sec.^{-1}) and at a pressure of 8 dn/cm as much as 4 times as high. The curve drawn in fig. 6 (curve I) is the mean of 12 curves found for different pH values; we have refrained from drawing them separately as with these rather inaccurate measurements an influence of the pH could not be detected. The latter observation corresponds with the compressibility data where no influence of pH was found either.

Tannic acid proved to exert an enormous influence on the viscosity. Above pressures of $1-1\frac{1}{2} \text{ dn/cm}$ the measurement could not be made with oscillations anymore owing to the great damping; at pressures of $2-3 \text{ dn/cm}$, however, the motion was aperiodic and slow enough to render it possible easily to measure the decrease of amplitude with time. The rigidity of the films also appears when enlarging the area of spread; then the film tears which is easy to observe when strewing talc powder on it. The films are very elastic. After torsion the angular velocity of the cylinder was high in the beginning but then decreased, sometimes very slowly, till the shearing velocity becomes almost constant. Also at high film pressures where, owing to the high viscosity of the film, the decrease in amplitude was small (e.g. 10%), this phenomenon occurred, which proves that it is not due to a change of shearing velocity with shearing stress but is caused by elastic recovery.

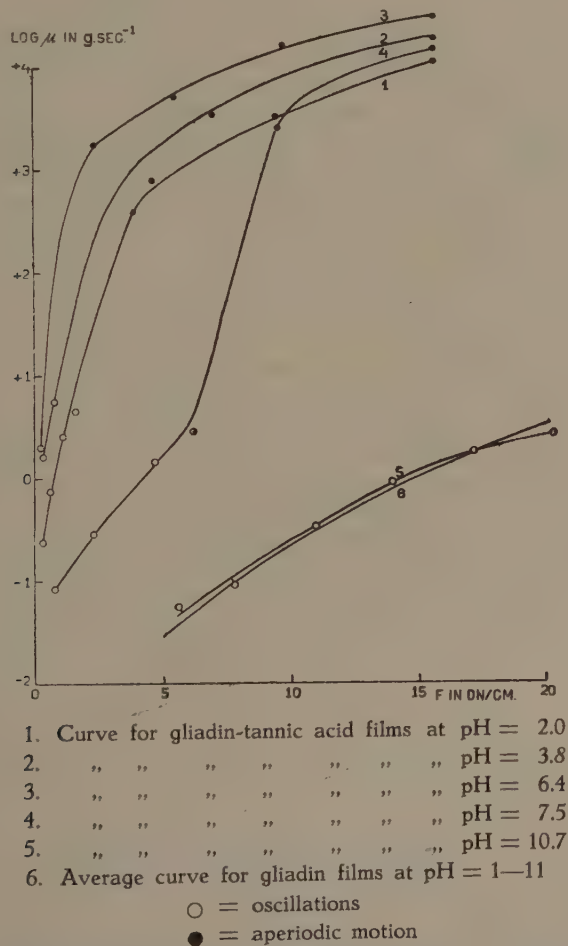


Fig. 7. Viscosity (logarithmic scale) of gliadin-tannic acid films at different surface pressures F .

In order to collect all viscosities in one graph, those of gliadin films as well as those of gliadin-tannic acid films, we have plotted the viscosities on a logarithmic scale against the pressure in fig. 7. The values (mean from two series of observations) obtained from measurements with oscillations (viscosities lower than about 8 g sec⁻¹) and those obtained from measurements with aperiodic motion (viscosities higher than about 200 g sec⁻¹) prove to lie on a pretty smooth curve. From the curves it is again evident that above pH = 7 the gliadin-tannic acid complex gradually passes into gliadin; at pH = 11 the influence of tannic acid has disappeared.

Finally it may be mentioned that trials to spread a preparation of the compound gliadin-tannic acid failed as the compound could not be dissolved in an indifferent solvent. The compound was prepared by mixing solutions of gliadin in aqueous ethylalcohol with an excess of tannic acid in the same solvent; the precipitated complex was rinsed with the solvent and dried in vacuo.

Summary.

The properties of gliadin and gliadin-tannic acid films were studied by measuring the surface pressure, the surface potential and the surface viscosity. In particular the influence of the pH of the underlying solutions were examined. It was very striking that all these measurements tally very well and lead to the same results, which may be summarized as follows:

Gliadin films are of the liquid type; above pressures of about 7 dn/cm gelation occurs. At higher pressures the film becomes plastic and elastic without getting solid, however. At pressures higher than about 20 dn/cm the film shows signs of collapse. At the lower pressures presumably free water molecules are squeezed out of the spaces in and between the micellae. At higher pressures either the orientation of the polypeptid chains changes or hydration water bound to the polar groups of the gliadin is pressed out of the film.

Maximum spreading of gliadin occurs in the vicinity of the isoelectric point (pH about 6.5); the maximum is much flatter than that for most other proteins. At a pH below about 4.5 and higher than about 8.5 the area of spread on very diluted buffers becomes small, but remains greater than in the case of most other proteins. The solvent alcohol may be the cause of these phenomena.

With respect to the influence of electrolytes on the spreading, gliadin behaves like other proteins. The pH has little influence on the character of gliadin films, but chiefly influences the amount of gliadin that collects in the surface.

Tannic acid strongly affects the character of gliadin films. Below pH = 7 tannic acid alters the film into a solid, very elastic one. At higher pH values the influence of tannic acid gradually decreases and at pH = 11 it has quite disappeared. On compressing tannic-acid gliadin films at pressures above 0.5 dn/cm phenomena occur resembling collapse; according to our measurements these phenomena must be ascribed to squeezing out of hydration water and real collapsing only occurs at pressures above 30–40 dn/cm.

University Hospital, Clinic of Pediatrics, Leiden.

Applied Mechanics. — *On the state of stress in perforated strips and plates.* (2nd communication.) By K. J. SCHULZ. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of March 28, 1942.)

4. *Cartesian representation of the stresses corresponding to the functions U (3, 14).* The stresses, corresponding to the stress functions (3, 14), can easily be expressed in polar coordinates by using (2, 4). However, the series obtained in this way have a restricted domain of convergence, which makes them unsuitable for the calculation of the stresses in the points of a line $z = c$ if c exceeds the limiting value b . Moreover, later on (see sect. 7) we shall need, particularly for points of such a line, cartesian expressions for the occurring stresses. Therefore we must construct other series, representing these stresses as Fourier series of the argument y and of the period b . To this end we refer to the function U_0 (3, 8), which in consequence of (3, 10) can be written as follows:

$$U_0 = \Re \left\{ \ln x + \sum_{k=1}^{\infty} \left[\ln \left(1 - \frac{x}{kb} \right) + \ln \left(1 + \frac{x}{kb} \right) \right] \right\} = \\ = \Re \left[\ln \frac{x}{b} \left(1 - \frac{x^2}{b^2} \right) \left(1 - \frac{x^2}{4b^2} \right) \left(1 - \frac{x^2}{9b^2} \right) \dots + \ln b \right]$$

if unessential constants are neglected. The expression between [] can be replaced by $\ln \sin \pi x/b - \ln \pi/b$, so that, with

$$\chi(x) = \ln \sin \pi x/b \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

U_0 may be represented by

$$U_0 = \Re \chi(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In order to deal with dimensionless numbers we introduce

$$\eta = 2\pi y/b, \quad \zeta = 2\pi z/b, \quad \xi = \eta + i\zeta = 2\pi(y + iz)/b = 2\pi x/b. \quad (3)$$

The function $\chi(x) = \ln \sin \xi/2$ can then be expanded into the following series

$$\chi(x) = \ln \left[-\frac{i}{2} \left(e^{\frac{i\xi}{2}} - e^{-\frac{i\xi}{2}} \right) \right] = \ln \left[\frac{ie^{\frac{-i\xi}{2}}}{2} (1 - e^{i\xi}) \right] = \\ = -\frac{1}{2} i\xi - \ln 2 + \frac{i\pi}{2} + \ln(1 - e^{i\xi}) = \\ = \ln \left[-\frac{ie^{\frac{i\xi}{2}}}{2} (1 - e^{-i\xi}) \right] = +\frac{1}{2} i\xi - \ln 2 - \frac{i\pi}{2} + \ln(1 - e^{-i\xi}).$$

Again neglecting unessential constants, and expanding $\ln(1-e^{i\zeta})$, resp. $\ln(1-e^{-i\zeta})$, into power series, we find the following two expressions

$$\chi(x) = \mp \frac{1}{2} i \zeta - \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm i n \zeta}, \quad (4)$$

which converge provided that $|e^{\pm i \zeta}| = e^{\mp \zeta} < 1$. This condition is satisfied — as can be controlled easily — if ζ resp. $z \geq 0$; therefore, the upper sign must be used if z is positive, the lower sign if z is negative. The series are divergent for $z = 0$. It follows from (2) and (4), that

$$U_0 = \pm \frac{1}{2} \zeta - \sum_{n=1}^{\infty} \frac{1}{n} e^{\mp n \zeta} \cos n \eta \quad (\text{for } z \geq 0);$$

consequently (see 3, 14)

$$U_{\sigma 0} = \pm \frac{1}{2} a^2 \zeta - a^2 \sum_{n=1}^{\infty} \frac{1}{n} e^{\mp n \zeta} \cos n \eta \quad \text{for } z \geq 0. \quad . . . (5)$$

Combination of (2, 1) and (3) leads to

$$\sigma_y = \frac{4\pi^2}{b^2} \frac{\partial^2 U_{\sigma 0}}{\partial \zeta^2}, \quad \sigma_z = \frac{4\pi^2}{b^2} \frac{\partial^2 U_{\sigma 0}}{\partial \eta^2}, \quad \tau_{yz} = -\frac{4\pi^2}{b^2} \frac{\partial^2 U_{\sigma 0}}{\partial \eta \partial \zeta}. \quad . . (6)$$

so that the following results are found for the stresses due to the function $U_{\sigma 0}$:

$$\left. \begin{aligned} \sigma_y &= -4\pi^2 \lambda^2 \sum_{n=1}^{\infty} n e^{\mp n \zeta} \cos n \eta, \\ \sigma_z &= +4\pi^2 \lambda^2 \sum_{n=1}^{\infty} n e^{\mp n \zeta} \cos n \eta, \\ \tau_{yz} &= \pm 4\pi^2 \lambda^2 \sum_{n=1}^{\infty} n e^{\mp n \zeta} \sin n \eta \end{aligned} \right\} \quad \text{for } z \geq 0, . . . (7)$$

Stresses in points of the y -axis are to be calculated with the aid of the expansions given in no. 3.

To obtain the stresses, corresponding to the other functions (3, 14), we make use of the formulae (3, 8) and (3, 10), from which we derive

$$\left. \begin{aligned} U_0 &= \operatorname{Re} \sum_{k=-\infty}^{k=+\infty} \ln(x-kb) = \operatorname{Re} \chi(x), \quad U_{2s} = \operatorname{Re} \sum_{k=-\infty}^{k=+\infty} (x-kb)^{-2s} \quad (s \geq 1), \\ \bar{U}_{2s+1} &= -\operatorname{Im} \sum_{k=-\infty}^{k=+\infty} (x-kb)^{-(2s+1)} \quad (s \geq 0), \\ U_{2s}^* &= \operatorname{Re} \sum_{k=-\infty}^{k=+\infty} (\bar{x}-kb)(x-kb)^{-2s+1}, \\ \bar{U}_{2s+1}^* &= -\operatorname{Im} \sum_{k=-\infty}^{k=+\infty} (\bar{x}-kb)(x-kb)^{-2s} \quad (s \geq 1). \end{aligned} \right\} (8)$$

It is seen at once that \bar{U}_{2s} and \bar{U}_{2s+1} are connected with U_0 (resp. with $\chi(x)$) by the relations

$$\left. \begin{aligned} U_{2s} &= -\frac{1}{(2s-1)!} \Re \frac{d^{(2s)} \chi(x)}{dx^{2s}} (s \equiv 1), \\ \bar{U}_{2s+1} &= -\frac{1}{(2s)!} \Im \frac{d^{(2s+1)} \chi(x)}{dx^{2s+1}} (s \equiv 0). \end{aligned} \right\} \quad (9)$$

Consequently they can be represented by the series

$$\left. \begin{aligned} U_{2s} &= \frac{(-1)^s (2\pi)^{2s}}{(2s-1)!} \frac{1}{b^{2s}} \sum_{n=1}^{\infty} n^{2s-1} e^{\mp n \zeta} \cos n \eta \quad (s \equiv 1), \\ \bar{U}_1 &= \pm \left(\frac{\pi}{b} + \frac{2\pi}{b} \sum_{n=1}^{\infty} e^{\mp n \zeta} \cos n \eta \right), \\ U_{2s+1} &= \pm \frac{(-1)^s (2\pi)^{2s+1}}{(2s)!} \frac{1}{b^{2s+1}} \sum_{n=1}^{\infty} n^{2s} e^{\mp n \zeta} \cos n \eta \quad (s \equiv 1) \end{aligned} \right\} \quad (10)$$

in which the ambiguous signs make a distinction between the two cases $z \geq 0$.

As regards the functions U_{2s}^* and \bar{U}_{2s+1}^* , we write

$$\begin{aligned} U_{2s}^* &= \Re \sum_{k=-\infty}^{k=+\infty} \frac{\bar{x} - kb}{(x - kb)^{2s-1}} = \Re \sum_{k=-\infty}^{k=+\infty} \frac{\bar{x} - x + x - kb}{(x - kb)^{2s-1}} = \\ &= \Re \left[\sum_{k=-\infty}^{k=+\infty} \frac{1}{(x - kb)^{2s-2}} - 2iz \sum_{k=-\infty}^{k=+\infty} \frac{1}{(x - kb)^{2s-1}} \right] \end{aligned}$$

or

$$U_{2s}^* = U_{2s-2} - 2z \bar{U}_{2s-1} \quad (s > 1) \text{ and likewise } \bar{U}_{2s+1}^* = \bar{U}_{2s-1}^* + 2z U_{2s} \quad (s \equiv 1). \quad (11)$$

U_2^* must be treated separately, but it is easily seen that

$$U_2^* = 1 - 2z U_1. \quad (11a)$$

From (10), (11) and (11a) it follows

$$\left. \begin{aligned} U_2^* &= 1 \mp \zeta \mp \sum_{n=1}^{\infty} 2\zeta e^{\mp n \zeta} \cos n \eta, \\ U_{2s}^* &= \frac{(-1)^s (2\pi)^{2s-2}}{(2s-2)!} \frac{1}{b^{2s-2}} \sum_{n=1}^{\infty} n^{2s-2} (\pm 2n \zeta - 2s + 2) e^{\mp n \zeta} \cos n \eta \quad (s > 1), \\ \bar{U}_{2s+1}^* &= \frac{(-1)^s (2\pi)^{2s-1}}{(2s-1)!} \frac{1}{b^{2s-1}} \sum_{n=1}^{\infty} n^{2s-2} [2n \zeta \mp (2s-1)] e^{\mp n \zeta} \cos n \eta \quad (z \leq 0). \end{aligned} \right\} \quad (12)$$

For completeness' sake it may be stated that \bar{U}_3^* differs from the general expression \bar{U}_{2s+1}^* with $(s=1)$ by the non-essential constant $\pm \pi/b$ for $z \geq 0$.

Substitution of (10) and (12) into (3, 14) provides us with the following series for

$$U_{\sigma, 2s}, U_{\tau, 2s}, \bar{U}_{\sigma, 2s+1}, \bar{U}_{\tau, 2s+1}:$$

$$\left. \begin{aligned} U_{\sigma, 2s} &= -\frac{1}{2} a^2 \frac{(-1)^s (2\pi\lambda)^{2s-2}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-3} \left[\frac{4\pi^2 \lambda^2 n^2}{2s+1} + 2s-2 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta, \\ U_{\tau, 2s} &= -\frac{1}{2} a^2 \frac{(-1)^s (2\pi\lambda)^{2s-2}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-3} \left[\frac{4\pi^2 \lambda^2 n^2 (2s+2)}{2s(2s+1)} + 2s-2 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta \end{aligned} \right\} \begin{matrix} (s > 1) \\ \text{for } z \geq 0 \end{matrix} \quad (13)$$

and

$$\left. \begin{aligned} \bar{U}_{\sigma 1} - \bar{U}_{\tau 1} &= \mp \frac{1}{2} a^2 (\pi\lambda + 2\pi\lambda \sum_{n=1}^{\infty} e^{\mp n\zeta} \cos n\eta), \\ \bar{U}_{\sigma, 2s+1} &= \pm \frac{1}{2} a^2 \frac{(-1)^s (2\pi\lambda)^{2s-1}}{(2s)!} \sum_{n=1}^{\infty} n^{2s-2} \left[\frac{4\pi^2 \lambda^2 n^2}{2s+2} + 2s-1 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta, \\ \bar{U}_{\tau, 2s+1} &= \pm \frac{1}{2} a^2 \frac{(-1)^s (2\pi\lambda)^{2s-1}}{(2s)!} \sum_{n=1}^{\infty} n^{2s-2} \left[\frac{4\pi^2 \lambda^2 n^2 (2s+3)}{(2s+1)(2s+2)} + 2s-1 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta \end{aligned} \right\} \begin{matrix} (s > 1) \\ \text{for } z \geq 0 \end{matrix} \quad (14)$$

The remaining functions $U_{\sigma 2}$ and $U_{\tau 2}$ can be derived from the general expressions $U_{\sigma, 2s}, U_{\tau, 2s}$ by adding $-\frac{1}{2} a^2 (1 \mp \zeta)$ or $+\frac{1}{2} a^2 (1 \mp \zeta)$ respectively, whereas $\bar{U}_{\sigma, 3}$ and $\bar{U}_{\tau, 3}$ can be obtained from $\bar{U}_{\sigma, 2s+1}$ and $\bar{U}_{\tau, 2s+1}$ by adding a term $\mp \frac{1}{4} a^2 \pi\lambda$.

We now are in a position to calculate the required stresses. Making use of (2, 1) the stresses corresponding to $U_{\sigma, 2s}$ ($s \geq 1$) are found to be for $z \geq 0$

$$\left. \begin{aligned} \sigma_y &= +\frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2}{2s+1} + 2s+2 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta, \\ \sigma_z &= -\frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2}{2s+1} + 2s-2 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta, \\ \tau_{yz} &= \mp \frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2}{2s+1} + 2s \mp 2n\zeta \right] e^{\pm n\zeta} \sin n\eta, \end{aligned} \right\} \begin{matrix} \text{those corresponding to } U_{\tau, 2s} (s \geq 1) \text{ for } z \geq 0 \\ \\ \end{matrix} \quad (15)$$

$$\left. \begin{aligned} \sigma_y &= -\frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2 (2s+2)}{2s(2s+1)} + 2s+2 \mp 2n\zeta \right] e^{\mp n\zeta} \cos n\eta, \\ \sigma_z &= +\frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2 (2s+2)}{2s(2s+1)} + 2s-2 \mp 2n\zeta \right] e^{\pm n\zeta} \cos n\eta, \\ \tau_{yz} &= \pm \frac{1}{2} \frac{(-1)^s (2\pi\lambda)^{2s}}{(2s-1)!} \sum_{n=1}^{\infty} n^{2s-1} \left[\frac{4\pi^2 \lambda^2 n^2 (2s+2)}{2s(2s+1)} + 2s \mp 2n\zeta \right] e^{\mp n\zeta} \sin n\eta \end{aligned} \right\}$$

in which the coefficients h, i, j, k are determined by

$$\left. \begin{aligned} h'_n &= j'_n = +4\pi^2 \lambda^2 n e^{-2\pi n \frac{\lambda}{\mu}}, \quad h'_n + i'_n = j'_n + k'_n = +4\pi^3 \lambda^3 n^2 e^{-2\pi n \frac{\lambda}{\mu}}, \\ h'_n &= -\frac{1}{2}(-1)^{\frac{s}{2}, \frac{s-1}{2}} \frac{(2\pi\lambda)^s}{(s-1)!} n^{s-1} \left(\frac{4\pi^2 \lambda^2 n^2}{s+1} + s - 2 - 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}}, \\ i'_n &= +\frac{1}{2}(-1)^{\frac{s}{2}, \frac{s-1}{2}} \frac{(2\pi\lambda)^s}{(s-1)!} n^{s-1} \left(\frac{4\pi^2 \lambda^2 n^2 (s+2)}{s(s+1)} + s - 2 - 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}}, \\ j'_n &= -\frac{1}{2}(-1)^{\frac{s}{2}, \frac{s-1}{2}} \frac{(2\pi\lambda)^s}{(s-1)!} n^{s-1} \left(\frac{4\pi^2 \lambda^2 n^2}{s+1} + s - 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}}, \\ k'_n &= +\frac{1}{2}(-1)^{\frac{s}{2}, \frac{s-1}{2}} \frac{(2\pi\lambda)^s}{(s-1)!} n^{s-1} \left(\frac{4\pi^2 \lambda^2 n^2 (s+2)}{s(s+1)} + s - 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}} (s \equiv 2). \end{aligned} \right\} \quad (19)$$

The notation $(-1)^{\frac{s}{2}, \frac{s-1}{2}}$ which occurs in the latter formulae must be understood in this sense, that even values of s require the factor $(-1)^{\frac{s}{2}}$, whereas odd values of s require the factor $(-1)^{\frac{s-1}{2}}$. As to the formulae (18) it should again be remembered that for $s=0$ the functions \bar{U}_{σ_1} and \bar{U}_{τ_1} only occur combined as $\bar{U}_{\sigma_1} - \bar{U}_{\tau_1}$. Finally it is obvious that, on account of the symmetry of the field, all stresses occurring in points of the line $z = -c$ likewise can be derived from (18). If we have to deal with the functions $U_{\sigma, 2s}, U_{\tau, 2s}$ the stresses σ_y, σ_z remain unchanged, whereas τ_{yz} changes its sign; if on the contrary we have to deal with the functions $\bar{U}_{\sigma, 2s+1}, \bar{U}_{\tau, 2s+1}$ the stresses σ_y and σ_z change their sign and τ_{yz} remains unchanged.

Mathematics. — *Bemerkung über die analytische Fortsetzung in bewerteten Körpern.*
 Von J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of March 28, 1942.)

Die Hauptprobleme der Analyse der bewerteten Körper sind von F. LOONSTRA in seiner Dissertation „Analytische Untersuchungen über bewertete Körper“ (Amsterdam 1941) untersucht worden. In Paragraph 12, Seite 34—39, untersucht er das spezielle Problem der analytischen Fortsetzung. Dieses Problem ist nur von Interesse für die nicht-archimedisch bewerteten Körper, da ein (nicht trivial) archimedisch bewerteter Körper bekanntlich isomorph ist zu einem mit gewöhnlichen Absolutbeträgen bewerteten Körper aus komplexen Zahlen. Beschränkt man das Problem also auf einem nicht-archimedisch bewerteten Körper (wobei man ohne der Allgemeinheit zu schaden und abgesehen von trivialen Fällen voraussetzen darf, dass die Charakteristik des Körpers Null ist, und der Primkörper p -adisch bewertet ist), so stellt sich heraus — im Gegensatz zur reellen und komplexen Funktionentheorie — dass die analytische Fortsetzung einer Potenzreihe unmöglich ist; das heisst: versucht man eine vorgegebene Potenzreihe $f(x)$ in einem Punkte x_0 ihres Konvergenzgebietes zu entwickeln in eine Potenzreihe nach aufsteigenden Potenzen von $x - x_0$, so konvergiert die neue Potenzreihe in *genau* demselben Gebiete wie die alte Potenzreihe.

Wir bringen in dieser Note einen zweiten kürzeren Beweis dieser von LOONSTRA bewiesener Tatsache. Vorher aber einige Bemerkungen.

Man beweist in einfacher Weise (siehe LOONSTRA, o.c., S. 10), dass eine Potenzreihe

$$f(x) = \sum_{n=0}^{\infty} a_n x^n (1)$$

definiert in einem nicht-archimedisch bewerteten Körper, dann und nur dann konvergiert, wenn

$$\lim_{n \rightarrow \infty} |a_n x^n| = 0.$$

Die Bewertung von a wird dabei mit $|a|$ bezeichnet.

Hieraus leitet man sofort ab, dass die zwei folgende prinzipiell verschiedene Fälle eintreten können: (1) konvergiert entweder für $|x| < R$, oder für $|x| \leq R$, wobei R eine passend gewählte reelle Zahl ist.

Nehmen wir in (1) die Bewertung jedes Gliedes

$$\sum_{n=0}^{\infty} |a_n| |x|^n (2)$$

und setzen wir $|x| = z$

$$g(z) = \sum_{n=0}^{\infty} |a_n| z^n (3)$$

so bekommt man eine Potenzreihe in z der gewöhnlichen reellen Funktionentheorie. Wir beweisen zuerst, dass der Konvergenzradius

$$\frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

dieser Reihe genau gleich R ist.

$f(x)$ konvergiert für x mit $|x| \leq R_1 < R$, also gilt

$$|a_n R_1^n| < \varepsilon \text{ für } n > n_0(\varepsilon)$$

oder

$$\sqrt[n]{|a_n|} < \frac{\sqrt[n]{\varepsilon}}{R_1}$$

also

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \frac{1}{R_1}$$

Der Konvergenzradius von $g(z)$ ist also mindestens gleich R_1 ; das gilt aber für alle $R_1 < R$, woraus die Behauptung folgt. Wir können dieses Resultat auch so ausdrücken: $f(x)$ konvergiert jedenfalls absolut für $|x| < R$.

Man bekommt also zwei Möglichkeiten:

1°. $f(x)$ konvergiert, und konvergiert absolut für $|x| < R$.

2°. $f(x)$ konvergiert für $|x| \leq R$, und konvergiert absolut für $|x| < R$.

In beiden Fällen ist

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

Auf diese zweite Möglichkeit (welche wir in LOONSTRA's Arbeit nicht antreffen) hat mich Herr H. FREUDENTHAL aufmerksam gemacht. Ein (von FREUDENTHAL herrührendes) Beispiel für diesen Fall ist die folgende Reihe:

$$f(x) = p + px + \dots + px^p + p^2 x^{p+1} + \dots + p^2 x^{p^2+p} + \\ + p^3 x^{p^2+p+1} + \dots + p^3 x^{p^3+p^2+p} + \dots$$

Man überzeugt sich leicht davon, dass diese Reihe konvergiert für $|x| \leq 1$, aber nur absolut konvergiert für $|x| < 1$.

Wir brauchen weiter die folgende Ungleichheit

$$\left| \frac{(n+k)!}{n! k!} \right| \leq \frac{(n+k)!}{n! k!} \cdot \dots \cdot \dots \quad (4)$$

n und k sind dabei natürliche Zahlen. Der Beweis dieser Ungleichheit ist leicht zu erbringen, um so mehr als wir sie nur brauchen für alle natürlichen Zahlen n von einer gewissen beliebigen Stelle n_0 an. Wir beweisen aber die schärfere Ungleichheit

$$\left| \frac{(n+k)!}{n! k!} \right| \leq 1 \cdot \dots \cdot \dots \quad (5)$$

dieselbe, welche LOONSTRA bei seinem Beweise benutzt. (5) ist eine unmittelbare Folge eines bekannten zahlentheoretischen Satzes (siehe z.B. E. LANDAU, Vorlesungen über Zahlentheorie, Band I, S. 13), welcher besagt, dass $l!$ (l ist eine beliebige natürliche Zahl) genau

$$\sum_{m=1}^{\infty} \left[\frac{l}{p^m} \right]$$

Primfaktoren p enthält, wenn $[\xi]$ die grösste ganze Zahl $\leq \xi$ ist. Die linke Seite von (5) enthält also

$$\sum_{m=1}^{\infty} \left[\frac{n+k}{p^m} \right] - \sum_{m=1}^{\infty} \left[\frac{n}{p^m} \right] - \sum_{m=1}^{\infty} \left[\frac{k}{p^m} \right]$$

Primfaktoren p . Wegen

$$\left\lfloor \frac{n+k}{p^i} \right\rfloor \geq \left\lfloor \frac{n}{p^i} \right\rfloor + \left\lfloor \frac{k}{p^i} \right\rfloor \quad (i=1, 2, \dots)$$

ist diese Anzahl ≥ 0 , und das heisst wegen der p -adischen Bewertung des Primkörpers, dasz (5) gilt.

Wir bringen jetzt den Beweis der Unmöglichkeit der analytischen Fortsetzung, zuerst aber für den Fall 1^o.

Versuchen wir die Potenzreihe (1) analytisch fortzusetzen durch Entwicklung in einem Punkte x_0 aus dem Gebiete $|x| < R$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \quad (6)$$

so konvergiert diese Reihe für genau dieselben Werte von x wie die Reihe

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(x_0)}{n!} \right| |x-x_0|^n \quad (7)$$

Wegen (4) gilt offenbar

$$\left| \frac{f^{(n)}(x)}{n!} \right| = \left| \sum_{k=0}^{\infty} \frac{(n+k)!}{n! k!} a_{n+k} x^k \right| \leq \sum_{k=0}^{\infty} \frac{(n+k)!}{n! k!} |a_{n+k}| |x|^k \quad (8)$$

Setzt man in der letzten Reihe aus (8) $|x| = z$, so bekommt man genau die Potenzreihe von $\frac{g^{(n)}(z)}{n!}$, welche bekanntlich jedenfalls konvergiert für alle Werte $\|z\| < R$, wenn wir für die gewöhnliche Absolutbetragbewertung Doppelstriche benutzen. Die Reihen aus (8) konvergieren deshalb gewiss für $|x| < R$, während ausserdem

$$\left| \frac{f^{(n)}(x)}{n!} \right| \leq \frac{g^{(n)}(z)}{n!} \quad (9)$$

gilt für alle $z = |x|$.

Die Entwicklung von $g(z)$ im Punkte $z_0 = |x_0|$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^n \quad (10)$$

konvergiert jedenfalls für $\|z-z_0\| < R$. Betrachten wir jene Werte von x , welche der Gleichung $|x-x_0| = \|z-z_0\| < R$ befriedigen, so ist — für einander zugehörigen Werte von x und z — (10), wegen (9), eine Majorante von (7). (7) und (6) konvergieren also für alle Werte x mit $|x-x_0| < R$. In unserm Körper sind aber die Werte von x mit $|x| < R$ und die Werte von x mit $|x-x_0| < R$ genau dieselbe. Hieraus geht hervor, dasz die Reihe (6) mindestens dasselbe Konvergenzgebiet hat wie die Reihe (1). Das Umgekehrte beweist man in genau derselben Weise, womit die Unmöglichkeit der analytischen Fortsetzung im Falle 1^o dargetan ist.

Setzen wir jetzt den Fall 2^o voraus, und nehmen wir an, dasz analytische Fortsetzung möglich wäre; dann würde (6) jedenfalls konvergieren für $|x| < R_2$, wobei R_2 eine passend gewählte Zahl grösser als R_1 bedeutet. Man beweist jetzt genau wie oben, dasz die Reihe (1) dann auch konvergieren würde für $|x| < R_2$ im Widerspruch mit dem gegebenen Divergenz dieser Reihe für $|x| > R_1$.

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4 .* (Zehnte Mitteilung.) Von W. J. BOS. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of March 28, 1942.)

Wir denken wieder $Q \neq 0$ und untersuchen hier zwei Büschel von invarianten Ebenen. Das erste Büschel liegt im Tangentialraume und das Zweite im „Beiraume“ mit der Gleichung $(B'x) = 0$.

§ 29.

Im Tangentialraume begeben wir die Fünfpunkteebene A mit der Gleichung (248):

$$(A'\pi)^2 = (5R - 6Q')\pi_{02,03} + 18Q \cdot \pi_{02,22} + 6Q \cdot \pi_{02,04} = 0$$

und die Oskulationsebene der Heftkurve im Punkte H in der Darstellung (239):

$$(h'\pi)^2 = (HH_1H_2\pi^2) = \frac{4}{27}(R - 6Q')\pi_{02,03} + \frac{4}{3}Q(\pi_{02,04} + \pi_{02,22}) = 0$$

Die q' - und λ -Gewichte der Komitanten $(A'\pi)^2$ und $(h'\pi)^2$ sind dieselben. Im Tangentialraume liegt also ein Büschel von invarianten Ebenen, welches gegeben werden kann durch:

$$\lambda(A'\pi)^2 + \mu(h'\pi)^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (250)$$

$\lambda = 3$, $\mu = -\frac{2}{3}$ gibt die Ebene:

$$(13R - 6Q')\pi_{02,03} + 36Q \cdot \pi_{02,22} = 0. \quad . \quad . \quad . \quad . \quad (251)$$

Für $\lambda = \frac{1}{3}$, $\mu = -\frac{2}{3}$ findet man die Ebene:

$$(R + 2Q')\pi_{02,03} - 4Q \cdot \pi_{02,04} = 0. \quad . \quad . \quad . \quad . \quad (252)$$

Mit Hilfe der Ebenen (251) und (252) bestimmen wir die Gleichung der Achse des Büschels (250). Die Ebene (251) können wir nämlich betrachten als die Schnittebene der Räume $x_{02} = 0$ und

$$(13R - 6Q')x_{03} + 36Q \cdot x_{22} = 0. \quad . \quad . \quad . \quad . \quad (253)$$

Die Achse des Büschels ist also die Schnittgerade der Ebene (252) mit dem Raume (253) und hat also die Gleichung:

$$-4Q(13R - 6Q')\pi_{03,02,04} + 36Q(R + 2Q')\pi_{22,02,03} - 4 \cdot 36Q^2 \cdot \pi_{22,02,04} = 0 \quad (254)$$

Nun ist:

$$\left. \begin{aligned} \pi_{03,02,04} &= -\pi_{02}\pi_{03}\pi_{04} = 2_{02}2_{03}\pi_{04} = (\pi^2 20^2)\pi_{04}2_{03} = \\ &= \frac{1}{3}(\pi^3 0^2)2_{04,03} = \frac{4}{3} \cdot 2_{03,13}(0^2\pi^3) \\ &= \frac{4}{3}Q(0^2\pi^3) \end{aligned} \right\} \quad (255)$$

Und:

$$\pi_{22,02,03} = -\pi_{02}\pi_{22}\pi_{03} = +0_{2\pi}0_{22}\pi_{03} = -0_{22}(\pi^2 02^2)\pi_{03} = -\frac{2}{3} \cdot 0_{22}(\pi^3 02)2_{03} = \left. \begin{aligned} &= \frac{2}{3}(\alpha^2 \pi^3) \end{aligned} \right\} \quad (256)$$

Weiter ist:

$$\begin{aligned} \pi_{22,02,04} &= 0_{2\pi}0_{22}\pi_{04} = -0_{22}(\pi^2 02^2)\pi_{04} = -\frac{2}{3} \cdot 0_{22}(\pi^3 02)2_{04} = \\ &= -\frac{2}{3} \cdot 0_{22}2_{04}(20\pi^3) = -\frac{2}{3} \cdot 0_{22}\dot{0}_{24}(\dot{0}0\pi^3) - \frac{2}{3} \cdot 0_{22}4_{02}(04\pi^3) \end{aligned}$$

Der erste Term gibt mit (1) und (104):

$$-\frac{2}{3} \cdot 0_{22}\dot{0}_{24}(\dot{0}0\pi^3) = -\frac{1}{3} \cdot 0_{22,24}(0^2\pi^3) = \frac{4}{9} \cdot 0_{13,24}(0^2\pi^3) = (-\frac{1}{2}\frac{0}{7}R + 4Q')(0^2\pi^3)$$

Also:

$$\pi_{22,02,04} = (-\frac{1}{2}\frac{0}{7}R + \frac{4}{9}Q')(0^2\pi^3) - \frac{2}{3} \cdot 0_{22}4_{02}(04\pi^3) \quad . \quad (257)$$

Mit Hilfe der Ausdrücke (255), (256) und (257) finden wir also für die Gleichung der Achse (254), nach Teilung durch $8Q$:

$$(R + 2Q')\{3(\alpha^2\pi^3) - 2Q(0^2\pi^3)\} + 12Q \cdot 0_{22}4_{02}(04\pi^3) = 0.$$

In Linienkoordinaten also:

$$(R + 2Q')(3a_{ik} - 4Q \cdot 0_{ik}) + 12Q \cdot 0_{22}4_{02}(04)_{ik} \quad . \quad (258)$$

Die Gleichung (256) resp. (255) gibt für die Schnittgerade der Ebene (251) resp. (252) mit der Heftebene die Gerade a_{ik} resp. 0_{ik} .

Die Ebene (251) ist also die Verbindungsebene der Achse (258) mit der Heftgerade a_{ik} und (252) ist die Ebene durch diese Achse und die Erzeugende 0_{ik} .

Das Ebenenbüschel (250) schneidet die Heftebene in dem Geradenbüschel:

$$(27\lambda + 2\mu)a_{ik} - (36\lambda + 8\mu)Q \cdot 0_{ik} \quad . \quad . \quad . \quad (259)$$

$\lambda = 0$, $\mu \neq 0$ gibt hier die Tangente h_{ik} der Heftkurve im Heftpunkte und $\lambda \neq 0$, $\mu = 0$ die Achse HG der Vierpunktebenen A . (Ich verbessere bei dieser Gelegenheit Gleichung (245), wo in der zweiten Zeile, 1 statt $\frac{1}{2}$ und $\frac{4}{3}$ statt $\frac{2}{3}$ zu stehen hat.)

Mittels einer ziemlich komplizierten Rechnung kann man zeigen, dass die Achse (258) des Büschels (250) den Punkt $H_{(2)}$ enthält, wo $H_{(2)}$ die zweite „kovariante Ableitung“ des Punktes H bedeutet. (Vgl. (160) und (163).)

§ 30.

Wir fanden im § 27: Die Vierpunktebenen schneiden die Beigerade g_{ik} . Auch die Fünfpunkteebene B mit der Gleichung

$$(B'\pi)^2 = (5R^2 - 6Q'R + 6QR' + 18QS)\pi_{02,03} + 18QR \cdot \pi_{02,22} - 36Q^2 \cdot \pi_{03,22} = 0$$

schneidet also g_{ik} in einem Punkte K . Im § 28 erhielten wir die Fünfpunkteebene B als Schnittebene der Räume $(B'x) = 0$ und $(a^2 4^2 x) = 0$. K ist also der Schnittpunkt von g_{ik} mit dem Raume $(a^2 4^2 x) = 0$. Mit Hilfe der Gleichungen (93) und (247) bekommen wir die Gleichung des Punktes K :

$$\left. \begin{aligned} K_{u'} &= (R^3 + 6QRR' - 6Q'R^2 + 9QRS - 18QQ'S - 18Q^2T)0_{22}0_{u'} + \\ &+ (-4R^2 + 24Q'R - 24QR' - 24QS)Q \cdot 2_{03}2_{u'} - 24(R' + 3S)Q^2 \cdot 0_{23}0_{u'} - \\ &- 4(5R - 6Q')Q^2 \cdot 0_{33}0_{u'} - 288Q^3 \cdot 2_{23}2_{u'} = 0. \end{aligned} \right\} \quad (260)$$

Der Beiraum enthält ausser der Fünfpunktebene B auch die Schnittebene V von zwei auf einanderfolgenden Beiräumen.

Nun ist:

$$(B'_1 x) = R' x_{02} + \left(\frac{2}{3} R - 2 Q'\right) x_{03} + \frac{2}{3} Q (x_{22} - x_{04}). \quad (261)$$

Die Gleichung der Ebene V lautet also:

$$(V' \pi)^2 = (\pi B') (\pi B'_1) = \left(\frac{2}{3} R^2 - 2 Q' R + 2 Q R'\right) \pi_{02,03} + \frac{2}{3} Q R (\pi_{02,22} - \pi_{02,04}) - \left\{ \begin{array}{l} - 3 Q^2 (\pi_{03,22} - \pi_{03,04}) = 0 \end{array} \right\} \quad (262)$$

Die Komitanten $(B' \pi)^2$ und $(V' \pi)^2$ haben gleiche φ' - und λ -Gewichte. Im Beiraume haben wir also ein Büschel von invarianten Ebenen, darstellbar durch:

$$\sigma (B' \pi)^2 + \tau (V' \pi)^2 = 0. \quad (263)$$

Wir behaupten dass die Gerade HK die Achse des Büschels ist und finden dann für diese Achse den Ausdruck:

$$\left. \begin{aligned} -\frac{1}{4Q} \cdot (HK)_{ik} &= (R^2 - 6 Q' R + 6 Q R' + 6 Q S) a_{ik} + \\ &+ \left(-\frac{10}{3} R^2 + 4 Q' R - 4 Q R' - 12 Q S\right) Q \cdot 0_{ik} + 72 Q^2 \cdot 0_{22} 2_{23} (02)_{ik}. \end{aligned} \right\} \quad (264)$$

Zum Beweise zeigen wir, dass die Ebene HGK , die Ebene also durch H und g_{ik} , im Büschel (263) enthalten ist.

Mit (93) bekommen wir für diese Ebene die Gleichung:

$$\left. \begin{aligned} (Hg^2 \pi^2) &= P \cdot 0_{22,0\pi} - 12 Q (3 Q S + R^2) 0_{22,1\pi} + \\ &+ 36 Q^2 R \cdot 0_{22,2\pi} - 72 Q^3 \cdot 0_{22,3\pi} = 0. \end{aligned} \right\} \quad (265)$$

Nun ist:

$$\begin{aligned} 0_{22,0\pi} &= 0; 0_{22,1\pi} = -2 \cdot 2_{02,1\pi} = +2 \cdot \pi_{02,12} = -\frac{2}{3} \pi_{02,03}; 0_{22,2\pi} = +\pi_{02,22}; \\ 0_{22,3\pi} &= -\frac{4}{3} \cdot 0_{13,3\pi} = +\frac{4}{3} \cdot \pi_{13,03} + \frac{4}{3} \cdot 3_{13,0\pi} = -\frac{4}{3} \cdot \pi_{03,13} - \frac{2}{3} \cdot 1_{33,0\pi} = \\ &= -\frac{4}{3} \cdot \pi_{03,13} + \frac{2}{3} \cdot 0_{33,1\pi}. \end{aligned}$$

Wegen der Identität (91) ist:

$$\begin{aligned} Q \cdot 0_{33,1\pi} &= 0_{13,23} \cdot 0_{33,1\pi} = +0_{13,33} \cdot 0_{23,1\pi} - 0_{23,33} \cdot 0_{13,1\pi} \\ &= R \cdot 0_{23,1\pi} - \frac{1}{2} \cdot S \cdot \pi_{02,03}. \end{aligned}$$

Dabei ist:

$$0_{23,1\pi} = -1_{23,0\pi} = +\frac{1}{2} \cdot 1_{14,0\pi} = +\frac{1}{4} \cdot 4_{02,0\pi} = -\frac{1}{4} \cdot \pi_{02,04}.$$

Damit wird

$$Q \cdot 0_{33,1\pi} = -\frac{1}{4} \cdot R \cdot \pi_{02,04} - \frac{1}{2} \cdot S \cdot \pi_{02,03}. \quad (266)$$

Die Gleichung der Ebene HGK wird also

$$\frac{1}{8Q}(Hg^2\pi^2) = (R^2 + 6QS)\pi_{02,03} + \frac{3}{2}QR(3\pi_{02,22} + \pi_{02,04}) + 12Q^2 \cdot \pi_{03,13} = 0,$$

oder wegen (1)

$$\frac{1}{8Q}(Hg^2\pi^2) = (R^2 + 6QS)\pi_{02,03} - 6QR \cdot \pi_{02,13} + 12Q^2 \cdot \pi_{03,13} = 0 \quad (267)$$

Diese Gleichung bekommen wir aus (263) wenn wir dort für σ und τ die Werte $\frac{1}{8}$ und -1 einsetzen; HK ist also die Achse des Büschels.

Das Ebenenbüschel (263) schneidet die Hefebene im Geradenbüschel:

$$(12\sigma + \tau)a_{ik} + 4\tau Q \cdot 0_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad (268)$$

$\sigma = 0, \tau \neq 0$ gibt die Gerade $m_{ik} = a_{ik} + 4Q \cdot 0_{ik}$.

Die Ebene V schneidet also die Hefebene in der Schnittgerade m_{ik} von drei aufeinanderfolgenden Tangentialräumen.

Wählen wir $\sigma = -\frac{1}{8}, \tau = 4$ dann gibt (263) die Gleichung der Ebene durch 0_{ik} und K :

$$(R^2 - 6Q'R + 6QR' + 6QS)\pi_{02,03} - 6QR \cdot \pi_{02,04} + 12Q^2 \cdot \pi_{03,04} = 0 \quad (269)$$

Mathematics. — Sur le théorème de MINKOWSKI, concernant un système de formes linéaires réelles. II. Deuxième communication: Lemmes et démonstration du théorème 1.
Par J. F. KOKSMA et B. MEULENBELD. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 28, 1942.)

§ 1. Dans la première communication nous avons posé le théorème 1 sans donner la démonstration. Nous voulons encore écrire ce théorème.

Théorème 1. Soient n et r des nombres naturels, $1 \leq r \leq n$; pour $\frac{n+1}{2} \leq r \leq n$ le nombre $Q_{n,r}$ soit désigné par

$$Q_{n,r} = 2^{n+1} \left\{ \frac{1}{(n+1)! r^r} \sum_{\mu=0}^r \binom{n+1}{\mu} \left(r - \frac{n+1}{2} \right)^\mu (n+1-r)^{r-\mu} + \right. \\ \left. + \frac{1}{r!} \sum_{\mu=0}^{n-r} \frac{r^{\mu+1}}{(n+1-r)^{\mu+1} (n-r-\mu)!} \left(\frac{n+1}{2r} \right)^{\frac{r(n-r-\mu)}{n+1-r}} P_\mu \right\}$$

où

$$P_\mu = \int_0^{\frac{2r-n-1}{n+1}} \frac{dv_{\mu+1}}{(v_{\mu+1}+1)^{\frac{n+1}{n+1-r}}} \int_0^{v_{\mu+1}} \frac{dv_\mu}{(v_\mu+1)^{\frac{n+1}{n+1-r}}} \cdots \int_0^{v_3} \frac{dv_2}{(v_2+1)^{\frac{n+1}{n+1-r}}} \int_0^{v_2} \frac{v_1^r dv_1}{(v_1+1)^{\frac{n+1}{n+1-r}}}$$

et pour $1 \leq r \leq n$ le nombre $Q_{n,r}^*$ par

$$Q_{n,r}^* = Q_{n,r} \quad , \quad \text{pour} \quad \frac{n+1}{2} \leq r \leq n,$$

$$Q_{n,r}^* = Q_{n,n+1-r} \quad , \quad \text{pour} \quad 1 \leq r < \frac{n+1}{2}.$$

En outre soient L_1, \dots, L_{n+1} des formes linéaires:

$$\begin{aligned} L_1 &= a_{11} x_1 + \dots + a_{1,n+1} x_{n+1}, \\ &\vdots \\ L_{n+1} &= a_{n+1,1} x_1 + \dots + a_{n+1,n+1} x_{n+1}, \end{aligned}$$

à coefficients réels $a_{\nu\mu}$, tels que le déterminant $\Delta = |a_{\nu\mu}|$ ne s'annule pas.

Alors à tout nombre $t > 2$ au moins un système de nombres entiers (x_1, \dots, x_{n+1}) correspond satisfaisant à

$$X = \max. (|x_1|, \dots, |x_{n+1}|) \leq 1,$$

et aux inégalités:

$$\sum_{\nu=1}^r |L_\nu| \leq 2 \sqrt[r]{\frac{t^{n+1-r} |\Delta|}{Q_{n,r}^*}}, \dots \dots \dots (1)$$

$$\sum_{v=r+1}^{n+1} |L_v| \leq \frac{2}{t}, \quad \dots \quad (2)$$

$$\left(\sum_{v=1}^r |L_v| \right)^r \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{n+1-r} \leq \frac{|\Delta|}{Q_{n,r}^*}, \quad \dots \quad (3)$$

$$|L_1 \dots L_{n+1}| \leq \frac{|\Delta|}{Q_{n,r}^* r^r (n+1-r)^{n+1-r}}. \quad \dots \quad (4)$$

Remarques.

1. L'inégalité (4) suit de (3). En vertu du théorème de la moyenne géométrique et la moyenne arithmétique nous avons d'après (3):

$$\begin{aligned} |L_1| \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{\frac{n+1-r}{r}} \cdot |L_2| \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{\frac{n+1-r}{r}} \dots |L_r| \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{\frac{n+1-r}{r}} \\ \leq \left(\frac{\left(\sum_{v=1}^r |L_v| \right) \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{\frac{n+1-r}{r}}}{r} \right)^r \leq \frac{|\Delta|}{Q_{n,r}^* r^r}, \end{aligned}$$

ou

$$|L_1 \dots L_r| \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{n+1-r} \leq \frac{|\Delta|}{Q_{n,r}^* r^r}. \quad \dots \quad (5)$$

Appliquant ce théorème encore nous obtenons d'après (5):

$$\begin{aligned} |L_1 \dots L_r|^{\frac{1}{n+1-r}} |L_{r+1}| \cdot |L_1 \dots L_r|^{\frac{1}{n+1-r}} |L_{r+2}| \dots |L_1 \dots L_r|^{\frac{1}{n+1-r}} |L_{n+1}| \\ \leq \left(\frac{|L_1 \dots L_r|^{\frac{1}{n+1-r}} \sum_{v=r+1}^{n+1} |L_v|}{n+1-r} \right)^{n+1-r} \leq \frac{|\Delta|}{Q_{n,r}^* r^r (n+1-r)^{n+1-r}}, \end{aligned}$$

d'où suit (4).

2. Pour la démonstration du théorème 1 (dans § 3 de cette communication) nous aurons besoin de quelques lemmes (les lemmes 1, 2 et 3 dans § 2), dont le plus essentiel est le lemme 1. La preuve du lemme 1 étant longue nous la donnerons dans une communication à part: la quatrième (et dernière) communication. La troisième communication contiendra la démonstration des lemmes 5 et 6 (cf § 4 de cette communication), dont nous aurons besoin chez la démonstration du lemme 1 dans la quatrième communication.

§ 2. **Lemme 1.** Soient $n, r, Q_{n,r}$ et $Q_{n,r}^*$ désignés comme en théorème 1, et soient L_1, \dots, L_{n+1} les $n+1$ formes:

$$\begin{aligned} L_1 &= b_{11} x_1, \\ L_2 &= b_{21} x_1 + b_{22} x_2, \\ &\vdots \\ L_{n+1} &= b_{n+1,1} x_1 + \dots + b_{n+1,n+1} x_{n+1}, \end{aligned}$$

à coefficients réels $b_{\nu\mu}$ ($b_{\nu\mu} = 0$ ($\mu > \nu$); $b_{\nu\nu} > 0$; $\nu, \mu = 1, \dots, n+1$) avec

$$b_{11} \dots b_{n+1,n+1} = \Delta.$$

Alors à tout nombre $t > 2$ au moins un système de nombres entiers (x_1, \dots, x_{n+1}) correspond satisfaisant à

$$X = \max. (|x_1|, \dots, |x_{n+1}|) \geq 1,$$

et aux inégalités (1), (2) et (3).

Remarque. Il suffira de démontrer ce lemme avec

$$\sum_{v=1}^r |L_v| < (2 + \delta) \sqrt[r]{\frac{t^{n+1-r} \Delta}{\varrho_{n,r}^*}}, \dots \quad (1a)$$

$$\sum_{v=r+1}^{n+1} |L_v| < \frac{2 + \delta}{t}, \dots \quad (2a)$$

$$\left(\sum_{v=1}^r |L_v| \right)^r \left(\sum_{v=r+1}^{n+1} |L_v| \right)^{n+1-r} < \frac{(1 + \delta)^{n+1} \Delta}{\varrho_{n,r}^*}, \dots \quad (3a)$$

(δ positif, arbitraire), au lieu des inégalités (1), (2) et (3), car puisque un nombre fini au plus de systèmes entiers (x_1, \dots, x_{n+1}) peuvent satisfaire à ces inégalités, le lemme suit par raison de continuité.

Ensuite nous utiliserons un théorème, dû à H. MINKOWSKI.⁹⁾ Citons le comme

Lemme 2. Soient L_1, \dots, L_{n+1} des formes linéaires de x_1, \dots, x_{n+1} :

$$\begin{aligned} L_1 &= a_{11} x_1 + \dots + a_{1,n+1} x_{n+1}, \\ &\vdots \\ L_{n+1} &= a_{n+1,1} x_1 + \dots + a_{n+1,n+1} x_{n+1}, \end{aligned}$$

à coefficients rationnels $a_{\nu\mu}$, tels que le déterminant $|a_{\nu\mu}|$ ne s'annule pas. Alors il existe toujours une substitution linéaire:

$$x_\nu = l_{\nu 1} y_1 + \dots + l_{\nu, n+1} y_{n+1} \quad (\nu = 1, \dots, n+1),$$

à coefficients entiers $l_{\nu\mu}$ et à déterminant $D = \pm 1$, telle que le système des formes L_1, \dots, L_{n+1} se transforme en un système de formes linéaires de y_1, \dots, y_{n+1} :

$$\begin{aligned} L'_1 &= b_{11} y_1, \\ L'_2 &= b_{21} y_1 + b_{22} y_2, \\ &\vdots \\ L'_{n+1} &= b_{n+1,1} y_1 + \dots + b_{n+1,n+1} y_{n+1}, \end{aligned}$$

dont les coefficients $b_{\nu\mu}$ jouissent des propriétés:

$$b_{\nu\mu} = 0 \ (\mu > \nu); \ b_{\nu\nu} > 0 \quad (\nu, \mu = 1, \dots, n+1). \quad (6)$$

Lemme 3. Soient $n, r, \varrho_{n,r}$ et $\varrho_{n,r}^*$ désignés comme en théorème 1, et soient L_1, \dots, L_{n+1} les $n+1$ formes linéaires:

$$\begin{aligned} L_1 &= a_{11} x_1 + \dots + a_{1,n+1} x_{n+1}, \\ &\vdots \\ L_{n+1} &= a_{n+1,1} x_1 + \dots + a_{n+1,n+1} x_{n+1}, \end{aligned}$$

à coefficients rationnels $a_{\nu\mu}$, tels que le déterminant $\Delta = |a_{\nu\mu}|$ ne s'annule pas.

⁹⁾ Voir: H. MINKOWSKI, Geometrie der Zahlen. Leipzig-Berlin, 1910, p. 65.

Alors à tout nombre réel $t > 2$ au moins un système de nombres entiers (x_1, \dots, x_{n+1}) correspond satisfaisant à

$$X = \max. (|x_1|, \dots, |x_{n+1}|) \geq 1,$$

et aux inégalités (1), (2) et (3).

Démonstration. En vertu du lemme précédent il existe une substitution linéaire :

$$x_\nu = l_{\nu 1} y_1 + \dots + l_{\nu, n+1} y_{n+1} \quad (\nu = 1, \dots, n+1),$$

à coefficients entiers $l_{\nu \mu}$ et à déterminant $D = \pm 1$, telle que L_1, \dots, L_{n+1} se transforment en

$$\left. \begin{aligned} L_1 &= b_{11} y_1, \\ L_2 &= b_{21} y_1 + b_{22} y_2, \\ &\vdots \\ L_{n+1} &= b_{n+1,1} y_1 + \dots + b_{n+1, n+1} y_{n+1}. \end{aligned} \right\} \dots \dots \dots (7)$$

avec (6).

D'après une propriété bien connue des substitutions linéaires il suit $b_{11} b_{22} \dots b_{n+1, n+1} = |\Delta|$.

Appliquons aux formes (7) le lemme 1. D'après ce lemme il y a au moins un système de nombres (y_1, \dots, y_{n+1}) avec

$$Y = \max. (|y_1|, \dots, |y_{n+1}|) \geq 1,$$

tel que les formes L_1, \dots, L_{n+1} en y_1, \dots, y_{n+1} vérifient les inégalités (1), (2) et (3). Comme les coefficients $l_{\nu \mu}$ sont entiers, les valeurs de x_1, \dots, x_{n+1} , qui correspondent à y_1, \dots, y_{n+1} sont entières aussi. Puisque $D = \pm 1$, il n'est pas possible que pour toutes les valeurs $\nu = 1, \dots, n+1$ vaut $x_\nu = 0$. Après cela on a (1), (2) et (3) avec $X \geq 1$, et le lemme est démontré.

§ 3. Démonstration du théorème 1.

Nous approximerons les coefficients réels $a_{\nu \mu}$ par des coefficients rationnels $a_{\nu \mu}$. Pour toute paire des nombres m et ε_m , m étant un nombre naturel et ε_m étant un nombre positif, nous pouvons trouver des nombres rationnels $a_{\nu \mu}$ avec $0 \leq a_{\nu \mu} - a_{\nu \mu} < \varepsilon_m$ ($\nu, \mu = 1, \dots, n+1$). Désignons par Δ_m le déterminant $|a_{\nu \mu}|$ et par $L_\nu^{(m)}$ ($\nu = 1, \dots, n+1$) la forme linéaire $a_{\nu 1} x_1 + \dots + a_{\nu, n+1} x_{n+1}$. Comme le déterminant est une fonction continue de ses éléments, on peut choisir pour tout nombre naturel $m = 1, 2, \dots$ le nombre ε_m assez petit, que l'inégalité

$$|\Delta - \Delta_m| < \frac{1}{m}$$

est satisfaite.

Aux formes $L_\nu^{(m)}$ ($\nu = 1, \dots, n+1$) nous appliquons le lemme 3. Il existe donc au moins un système de nombres (x_1, \dots, x_{n+1}) , dépendant de m , avec

$$X = \max. (|x_1|, \dots, |x_{n+1}|) \geq 1,$$

qui vérifie les inégalités :

$$\sum_{\nu=1}^r |L_\nu^{(m)}| \leq 2 \sqrt[r]{\frac{t^{n+1-r} |\Delta_m|}{\varrho_{n,r}^*}}, \quad \dots \dots \dots (8)$$

$$\sum_{v=r+1}^{n+1} |L_v^{(m)}| \leq \frac{2}{t}, \quad \dots \quad (9)$$

$$\left(\sum_{v=1}^r |L_v^{(m)}| \right)^r \left(\sum_{v=r+1}^{n+1} |L_v^{(m)}| \right)^{n+1-r} \leq \frac{|\Delta_m|}{Q_{n,r}^*} \quad \dots \quad (10)$$

Des formules $|\Delta_m| \leq |\Delta| + 1$, (8) et (9) nous déduisons :

$$|L_v^{(m)}| < R \quad (v = 1, \dots, n+1),$$

où R est un nombre qui ne dépend pas de m . Quand m parcourt la suite des nombres naturels, nous aurons donc au plus un nombre fini de points (x_1, \dots, x_{n+1}) , vérifiant les inégalités (8), (9) et (10). Ainsi au moins un de ces points (x_1, \dots, x_{n+1}) avec $X \geq 1$ pour une suite infinie de nombres croissants m_k ($k = 1, 2, \dots$) vérifie les inégalités :

$$\sum_{v=1}^r |L_v^{(m_k)}| \leq 2 \sqrt[r]{\frac{t^{n+1-r} |\Delta_{m_k}|}{Q_{n,r}^*}},$$

$$\sum_{v=r+1}^{n+1} |L_v^{(m_k)}| \leq \frac{2}{t},$$

$$\left(\sum_{v=1}^r |L_v^{(m_k)}| \right)^r \left(\sum_{v=r+1}^{n+1} |L_v^{(m_k)}| \right)^{n+1-r} \leq \frac{|\Delta_{m_k}|}{Q_{n,r}^*}.$$

Si k croît indéfiniment, on a $a_{v\mu} \rightarrow a_{v\mu}$; comme les formes linéaires sont continues à l'égard des $a_{v\mu}$ et comme le point (x_1, \dots, x_{n+1}) est fixe, on a

$$L_v^{(m_k)} \rightarrow L_v \quad (v = 1, \dots, n+1), \quad \Delta_{m_k} \rightarrow \Delta.$$

Ainsi le théorème est démontré.

§ 4. La démonstration du lemme 1 repose sur un théorème dû à M. H. F. BLICHFELDT¹⁰). Nous le citons comme

Lemme 4. L'espace R_m à m dimensions ($m \geq 2$) des points (u_1, \dots, u_m) soit divisé par les "plans" :

$$u_v = a_v + b_v t \quad (v = 1, 2, \dots, m; t = 0, \pm 1, \pm 2, \dots; a_v, b_v \text{ fixes})$$

en parallélépipèdes R . Dans chaque R_k ($k \geq 1$) points arbitraires fixes soient choisis, se nommant ici les points spéciaux de R . Le volume de R soit W . Si S est un ensemble borné, arbitraire, ouvert et continument lié, à tout $\varepsilon > 0$ une translation correspond, par laquelle l'ensemble S est transféré dans une telle position, que le nombre des points spéciaux de R , qui se trouvent à l'intérieur de S ou à l'intérieur d'une sphère de rayon ε et ayant un point-frontière de S pour centre, est supérieur à $\frac{V_k}{W}$.

En outre nous aurons besoin des lemmes suivants.

Lemme 5. Soient n et r désignés comme en théorème 1, A et B des nombres réels, arbitraires, et pour $0 \leq \sigma \leq n - r$ la forme D_σ définie par

$$D_\sigma = B - \sum_{v=n+2-\sigma}^{n+1} u_v$$

(une somme vide soit égale à zéro).

¹⁰) Voir: M. H. F. BLICHFELDT, A new principle in the geometry of numbers, with some applications. Trans. Amer. Soc. 15, (1914), p. 227—235.

Alors l'intégrale plurielle

$$J = \frac{1}{r!} \int_0^{D_0} du_{n+1} \int_0^{D_1} du_n \dots \int_0^{D_{n-r}} \left(1 - A \sum_{v=r+1}^{n+1} u_v \right)^r du_{r+1}$$

est égale à

$$\frac{1}{(n+1)!} \left\{ \frac{1}{A^{n+1-r}} - \sum_{k=0}^{n-r} \binom{n+1}{k} \frac{B^k (1-AB)^{n+1-k}}{A^{n+1-r-k}} \right\}.$$

Lemme 6. Soient n et r des nombres naturels avec $\frac{n+1}{2} \leq r \leq n$, soient $q_{n,r}$ et P_μ définis comme en théorème 1 et soient t et a des nombres positifs. Dans l'espace R_{n+1} à $n+1$ dimensions des points (u_1, \dots, u_{n+1}) l'ensemble ouvert S soit défini par

$$\sum_{v=r+1}^{n+1} |u_v| t < 1,$$

$$\sum_{v=1}^r \frac{|u_v|}{a} + \frac{2^{\frac{n+1}{n+1-r}} r^{\frac{r}{n+1-r}} (n+1-r) \sum_{v=r+1}^{n+1} |u_v| t}{(n+1)^{\frac{n+1}{n+1-r}}} < 1, \text{ si } \sum_{v=r+1}^{n+1} |u_v| t \leq \left(\frac{n+1}{2r} \right)^{\frac{r}{n+1-r}},$$

$$\sum_{v=r+1}^{n+1} |u_v| t \left\{ \sum_{v=1}^r \frac{|u_v|}{a} + 1 \right\}^{\frac{r}{n+1-r}} < 1, \text{ si } \left(\frac{n+1}{2r} \right)^{\frac{r}{n+1-r}} < \sum_{v=r+1}^{n+1} |u_v| t < 1.$$

Alors le volume V de S est égal à

$$\frac{a^r q_{n,r}}{t^{n+1-r}}.$$

Mathematics. — *Die Begründung der Trigonometrie in der hyperbolischen Ebene.* (Erste Mitteilung.) Von J. C. H. GERRETSEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 28, 1942.)

Einleitung.

Von H. LIEBMANN ¹⁾ ist der Versuch gemacht worden, die Trigonometrie in der hyperbolischen Ebene mittels der von D. HILBERT ²⁾ entwickelten Endenrechnung zu begründen. Wenn man die Forderung aufrecht erhalten will, dass keine Stetigkeitsannahmen gemacht werden dürfen, muss dieser Versuch als misslungen betrachtet werden. Denn an entscheidender Stelle werden Stetigkeitsaxiome benutzt und es wird dabei nicht hervorgehoben inwiefern sie vermieden werden können. Ausserdem sind die Betrachtungen LIEBMANNs wenig elegant, da für die Herleitung der Formeln der Längenmessung eine ziemlich verwickelte Hilfsfunktion eingeführt wird, deren geometrische Bedeutung nicht sofort einleuchtet.

Die Frage nach der Begründung der bekannten trigonometrischen Formeln ist besonders interessant im Lichte einer Aeusserung von FR. SCHUR ³⁾:

„.....Es scheint mir daher nicht ganz verständlich, welche Ableitung Herr Hilbert im Auge hat, wenn er am Schlusse seiner Abhandlung sagt: „Auch sind dann die bekannten Formeln der Bolyai-Lobatschewskijschen Geometrie ohne Schwierigkeit ableitbar“. Denn alle mir sonst bekannten Formeln der nicht-Euklidischen Geometrie — und nur diese können doch gemeint sein — und auch ihre Ableitungen enthalten die Exponentialfunktion, können also bei Vermeidung eines Stetigkeitsaxioms nicht in Betracht kommen“.

Ich möchte zeigen, dass dieser Skeptizismus unberechtigt ist. Tatsächlich kann der ganze Formelapparat der hyperbolischen Trigonometrie aufgebaut werden, wobei nur die in der HILBERTschen Abhandlung aufgezählten Axiome benutzt werden. Die Axiome der ersten drei Gruppen werden wir als die *Axiome der absoluten ebenen Geometrie* bezeichnen, während das Axiom der vierten Gruppe das HILBERTsche *Parallelenaxiom* heissen möge.

Ich setze im Folgenden die Bekanntschaft mit den wichtigsten Sätzen der elementaren hyperbolischen Geometrie voraus; man kann sie in der HILBERTschen Abhandlung nachlesen. Aber die Endenrechnung werde ich zum besseren Verständnis der sich darauf stützenden Betrachtungen aufs neue herleiten in einer Weise, welche einigermaßen von der von HILBERT gegebenen abweicht. Darauf werde ich die Bewegungen der hyperbolischen Ebene betrachten, womit die Grundlage für die Erklärung der hyperbolischen und trigonometrischen Funktionen geschaffen wird. In einfachster Weise werde ich weiter die berühmte LOBATSCHESKIJsche Formel hinsichtlich des Parallelwinkels herleiten.

Eigentlich wäre damit das Ziel erreicht, denn schon LOBATSCHESKIJ ⁴⁾ hat ein Verfahren angegeben, womit sämtliche Formeln des rechtwinkligen Dreiecks gefunden

1) H. LIEBMANN, Ueber die Begründung der hyperbolischen Geometrie, Math. Ann. 59, 110—128 (1904).

2) D. HILBERT, Neue Begründung der Bolyai-Lobatschewskijschen Geometrie, Math. Ann. 57, 137—150 (1903). Wieder abgedruckt in: Grundlagen der Geometrie, 7 Aufl. (Leipzig und Berlin 1930), Anhang III.

3) FR. SCHUR, Zur Bolyai-Lobatschewskijschen Geometrie, Math. Ann. 59, 314—320 (1904).

4) N. I. LOBATSCHESKIJ, Zwei geometrische Abhandlungen. Uebersetzt und mit Anmerkungen versehen von F. ENGEL (Leipzig 1899), S. 20

werden können; dieses Verfahren ist in mehr oder wenig abgeänderter Form in der Literatur mehrmals benutzt worden⁵⁾. Ich werde aber diese Formeln unmittelbar mit Benutzung der Endenrechnung herleiten.

Nebenbei kann noch eine interessante Frage beantwortet werden. Schon FR. SCHUR⁶⁾ hat bemerkt, dass das HILBERTsche Parallelenaxiom ein besonderes Axiom über das Schneiden eines Kreises mit einer Geraden, die einen Punkt innerhalb des Kreises enthält, entbehrlich macht. Dieser Satz hängt aufs engste mit der Parallelenkonstruktion zusammen. Man hat die Vermutung ausgesprochen⁷⁾, dass ein etwaiges Vorhandensein der Schnittpunkte zweier Kreise nicht ohne das Axiom der Stetigkeit oder das ihm gleichwertige Archimedische Axiom, zu dem ausserdem das HILBERTsche Vollständigkeitsaxiom hinzutreten muss, beweisbar ist. Ich werde aber zeigen, dass diese Vermutung falsch ist, indem ich mit Hilfe der Trigonometrie einen ganz einfachen Beweis dieses Satzes gebe. Der erstgenannte Satz lässt sich auch sehr einfach trigonometrisch beweisen.

§ 1. Die Endenrechnung.

Wenn die Halbgeraden AA' und BB' nicht zu einer einzigen Geraden gehören, möge die erste zu der zweiten *parallel* genannt werden, wenn die Halbgeraden keinen Punkt gemeinsam haben und jede von A ausgehende innerhalb des Winkels $A'AB$ liegende Halbgerade die Halbgerade BB' trifft. Wenn die Halbgeraden AA' und BB' jedoch zu einer einzigen Geraden gehören, mögen sie untereinander *parallel* genannt werden, wenn die Punkte von einer der Halbgeraden sämtlich zu der anderen Halbgeraden gehören.

Unter alleiniger Benutzung der Axiome der absoluten ebenen Geometrie kann man zeigen:

1. *Mit der Halbgeraden AA' ist auch jede Halbgerade, welche zu der Geraden AA' gehört und mit der ersten Halbgeraden parallel ist, parallel zu der Halbgeraden BB' (Erhaltung des Parallelismus längs einer Geraden).*

2. *Wenn die Halbgerade AA' zu der Halbgeraden BB' parallel ist, so ist auch diese zu jener parallel (Gegenseitigkeit des Parallelismus).*

3. *Wenn die Halbgeraden AA' und BB' zu der Halbgeraden CC' parallel sind, so sind AA' und BB' untereinander parallel (Fortpflanzung des Parallelismus).*

Von allen Halbgeraden, die zu einander parallel sind, sagt man, dass sie dasselbe *Ende* bestimmen; durch nicht-parallele Halbgeraden werden verschiedene Enden bestimmt.

Wir werden sagen, dass die Gerade g „durch“ das Ende α „geht“, oder dass α „auf“ g „liegt“, oder dass g das Ende α mit einem Punkt oder mit einem anderen Ende „verbindet“, wenn eine Halbgerade zu g gehört, die das Ende α bestimmt. Ist A ein Punkt auf der Geraden g , dann werden wir diese Gerade mit $(A\alpha)$ bezeichnen. Offenbar liegen auf einer Geraden immer zwei Enden.

Das *Parallelenaxiom* von HILBERT besagt im wesentlichen folgendes:

4. *Jedes Ende kann mit einem Punkt durch eine und nur eine Gerade verbunden werden. Durch zwei Enden geht höchstens eine Gerade.*

Wenn man das Parallelenaxiom zu den Axiomen der absoluten Geometrie hinzunimmt, kann man beweisen:

5. *Durch irgend zwei Enden geht immer eine Gerade.*

⁵⁾ H. LIEBMANN, Elementargeometrischer Beweis der Parallelenkonstruktion und neue Begründung der trigonometrischen Formeln der hyperbolischen Geometrie, Math. Ann. **61**, 185—199 (1905).

R. BONOLA, Sulla teoria delle parallele e sulle geometrie non-euclidee, Questioni riguardanti le matematiche elementari, I₂, 3a Ed. (Bologna 1925), § 36.

⁶⁾ a.a.O. S. 319.

⁷⁾ M. SIMON, K. FLADT, Nichteuclidische Geometrie (Leipzig und Berlin 1925), S. 54.

Die durch die Enden α und β gehende Gerade wird mit (α, β) bezeichnet.

Aus vorigem Satz in Verbindung mit dem Parallelenaxiom geht leicht hervor:

6. Wenn irgend eine Gerade g und ein Ende α , das nicht auf der Geraden liegt, vorgelegt sind, dann gibt es eine und nur eine Gerade durch α , die auf der Geraden g senkrecht steht.

Denn die Gerade, welche α mit dem Spiegelbild α' von α in bezug auf die Gerade g verbindet, steht auf g senkrecht. Es gibt durch α kein zweites Lot auf g , denn dieses müsste auch durch α' gehen.

Wir werden sehr oft mit Bewegungen zu tun haben. Unter einer Bewegung werde eine Transformation der Ebene in sich verstanden, wobei irgend einem Dreieck ein damit kongruentes Dreieck zugeordnet wird. Wenn \mathfrak{B}_1 und \mathfrak{B}_2 zwei Bewegungen sind, so ist die Transformation, welche entsteht, wenn man zuerst \mathfrak{B}_1 und dann \mathfrak{B}_2 ausübt, ebenfalls eine Bewegung. Sie heisst das *Produkt* der Bewegungen \mathfrak{B}_1 und \mathfrak{B}_2 und wird mit $\mathfrak{B}_2 \mathfrak{B}_1$ bezeichnet; dabei hat man auf die Reihenfolge der Faktoren zu achten. Wegen $(\mathfrak{B}_3 \mathfrak{B}_2) \mathfrak{B}_1 = \mathfrak{B}_3 (\mathfrak{B}_2 \mathfrak{B}_1)$, sind Klammern bei der Produktbildung von mehr als zwei Faktoren überflüssig.

Bei einer Spiegelung an einer Geraden wird irgend einem Punkt P der mit ihm in bezug auf die Gerade symmetrisch gelegene Punkt P' zugeordnet. Dabei bleiben die Punkte der Geraden ungeändert. Eine Spiegelung ist eine Bewegung, wie aus den Kongruenzsätzen leicht hervorgeht. Ausserdem erkennen wir:

7. Eine Spiegelung ist eine involutorische Transformation; d.h. das Produkt zweier Spiegelungen an der nämlichen Geraden ist eine Transformation, welche jedem Punkt sich selbst zuordnet.

Man sieht ohne Mühe ein, dass durch eine Bewegung eine umkehrbar eindeutige Zuordnung der Enden untereinander hervorgerufen wird.

Den weiteren Betrachtungen legen wir zunächst folgenden Satz zugrunde:

8. Wenn a , b und c drei Geraden sind, welche durch das nämliche Ende ω gehen und die Spiegelungen an diesen Geraden bezw. mit \mathfrak{S}_a , \mathfrak{S}_b und \mathfrak{S}_c bezeichnet werden, so gibt es stets eine eindeutig bestimmte Gerade d durch dasselbe Ende ω , so dass das Produkt der Spiegelungen an den Geraden a , b und c der Spiegelung \mathfrak{S}_d an der Geraden d gleichkommt.

In einer Formel ausgedrückt:

$$\mathfrak{S}_c \mathfrak{S}_b \mathfrak{S}_a = \mathfrak{S}_d. \quad (1, 1)$$

Den Beweis lese man bei HILBERT nach.

Wir fügen noch hinzu:

9. Wenn die Geraden a , b und c den im vorigen Satz genannten Voraussetzungen genügen, dann kommt das Produkt der Spiegelungen an den Geraden a , b und c dem Produkt der Spiegelungen an den Geraden c , b und a gleich.

Also:

$$\mathfrak{S}_c \mathfrak{S}_b \mathfrak{S}_a = \mathfrak{S}_a \mathfrak{S}_b \mathfrak{S}_c. \quad (1, 2)$$

Die Richtigkeit dieser Behauptung erhellt sofort aus der Tatsache, dass eine Spiegelung eine involutorische Transformation ist.

Für eine spätere Anwendung erwähnen wir folgenden, übrigens von HILBERT nicht genannten Satz:

10. Wenn a , b und c drei Geraden sind, welche auf der nämlichen Geraden g senkrecht stehen und die Spiegelungen an jenen Geraden bezw. mit \mathfrak{P}_a , \mathfrak{P}_b und \mathfrak{P}_c bezeichnet werden, so gibt es stets eine eindeutig bestimmte Gerade d , senkrecht auf derselben Geraden g , so dass das Produkt der Spiegelungen an den Geraden a , b und c der Spiegelung \mathfrak{P}_d an der Geraden d gleichkommt.

In der Formel:

$$\mathfrak{P}_c \mathfrak{P}_b \mathfrak{P}_a = \mathfrak{P}_d. \quad (1, 3)$$

Der Beweis dieses Satzes sieht dem Beweis des Satzes 8 ganz ähnlich.

Dieser Satz kann wieder ergänzt werden mit:

11. Wenn die Geraden a , b und c den im vorigen Satz genannten Voraussetzungen genügen, dann kommt das Produkt der Spiegelungen an den Geraden a , b und c dem Produkt der Spiegelungen an den Geraden c , b und a gleich.

Also:

$$\mathfrak{P}_c \mathfrak{P}_b \mathfrak{P}_a = \mathfrak{P}_a \mathfrak{P}_b \mathfrak{P}_c. \quad (1, 4)$$

Wir wollen nun Verknüpfungen der Enden erklären und die Rechnungsregeln herleiten.

a. Die Addition der Enden. Es werde eine bestimmte Gerade angenommen und deren Enden mit 0 und ∞ bezeichnet. Die Menge der von ∞ verschiedenen Enden werde \mathfrak{h} genannt. Die Spiegelung an der Geraden (ξ, ∞) , wobei ξ irgend ein Element von \mathfrak{h} ist, lässt die Enden ξ und ∞ ungeändert; wir werden diese Spiegelung mit \mathfrak{S}_ξ bezeichnen.

Es seien α und β irgend zwei Elemente von \mathfrak{h} . Auf Grund des Satzes 8 gibt es eine eindeutig bestimmte Gerade (σ, ∞) derart, dass für die Spiegelung \mathfrak{S}_σ an dieser Geraden die Beziehung

$$\mathfrak{S}_\sigma = \mathfrak{S}_\beta \mathfrak{S}_0 \mathfrak{S}_\alpha. \quad (1, 5)$$

erfüllt ist. Das Element σ heisst die Summe $\alpha + \beta$ der Enden α und β .

12. Es gilt das kommutative Gesetz:

$$\alpha + \beta = \beta + \alpha. \quad (1, 6)$$

Denn wegen der Formeln (1,5) und (1,2) haben wir:

$$\mathfrak{S}_{\alpha+\beta} = \mathfrak{S}_\beta \mathfrak{S}_0 \mathfrak{S}_\alpha = \mathfrak{S}_\alpha \mathfrak{S}_0 \mathfrak{S}_\beta = \mathfrak{S}_{\beta+\alpha}.$$

13. Es gilt das assoziative Gesetz:

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma. \quad (1, 7)$$

Denn wir haben:

$$\begin{aligned} \mathfrak{S}_{\alpha+(\beta+\gamma)} &= \mathfrak{S}_{\beta+\gamma} \mathfrak{S}_0 \mathfrak{S}_\alpha = \mathfrak{S}_\gamma \mathfrak{S}_0 \mathfrak{S}_\beta \mathfrak{S}_0 \mathfrak{S}_\alpha \\ &= \mathfrak{S}_\gamma \mathfrak{S}_0 \mathfrak{S}_{\alpha+\beta} = \mathfrak{S}_{(\alpha+\beta)+\gamma}. \end{aligned}$$

14. Es gibt in \mathfrak{h} ein neutrales Element, die Null:

$$\alpha + 0 = \alpha, \quad (1, 8)$$

für jedes Element α von \mathfrak{h} .

Denn:

$$\mathfrak{S}_{\alpha+0} = \mathfrak{S}_0 \mathfrak{S}_0 \mathfrak{S}_\alpha = \mathfrak{S}_\alpha.$$

Das Spiegelbild des Endes α in bezug auf die Gerade $(0, \infty)$ wird mit $-\alpha$ bezeichnet und heisst das Entgegengesetzte von α . Offenbar ist $-(-\alpha) = \alpha$.

15. Die Summe irgend eines Elements von \mathfrak{h} und seines Entgegengesetzten ist Null:

$$\alpha + (-\alpha) = 0, \quad (1, 9)$$

für jedes α von \mathfrak{h} .

Denn, man prüft leicht nach, dass die Spiegelung $\mathfrak{S}_{\alpha+(-\alpha)} = \mathfrak{S}_{-\alpha} \mathfrak{S}_0 \mathfrak{S}_\alpha$ die Enden α und $-\alpha$ mit einander vertauscht und also die Spiegelung \mathfrak{S}_0 sein muss.

Die gefundenen Ergebnisse können folgendermassen zusammengefasst werden:

16. Die Elemente von \mathfrak{h} bilden gegenüber der Addition eine Abelsche Gruppe.

Daraus folgt:

17. Sind α und β irgend zwei Elemente aus \mathfrak{h} , so gibt es immer ein eindeutig bestimmtes Element ξ aus \mathfrak{h} , das der Gleichung

$$\beta + \xi = \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 10)$$

genügt.

Die Lösung wird mit $\alpha - \beta$ bezeichnet und ist gleich $\alpha + (-\beta)$; insbesondere ist $0 - \beta = -\beta$. Wir nennen $\alpha - \beta$ die *Differenz* der Enden α und β .

Für die späteren Entwicklungen ist folgender Satz von hervorragender Bedeutung:

18. Ist α ein Element von \mathfrak{h} , so wird durch die Bewegung $\mathfrak{S}_\alpha \mathfrak{S}_0$ dem willkürlichen Ende ξ das Ende $\xi + 2\alpha$ vermöge:

$$\xi = \xi + 2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 11)$$

zugeordnet.

Selbstverständlich wird unter 2α die Summe $\alpha + \alpha$ verstanden. Es sei nun ξ ein Element von \mathfrak{h} und $\bar{\xi}$ das aus $-\xi$ durch Spiegelung an der Geraden (α, ∞) hervorgehende Ende. Die Transformation $\mathfrak{S}_\alpha \mathfrak{S}_0 \mathfrak{S}_\xi \mathfrak{S}_0 \mathfrak{S}_\alpha = \mathfrak{S}_{\xi+2\alpha}$ ist einerseits eine Spiegelung an der Geraden $(\xi + 2\alpha, \infty)$, lässt aber anderseits die durch die Bewegung $\mathfrak{S}_\alpha \mathfrak{S}_0$ aus der Geraden (ξ, ∞) hervorgehende Gerade $(\bar{\xi}, \infty)$ ungeändert. Also musz $\xi = \xi + 2\alpha$ sein. Der Satz gilt auch für $\xi = \infty$, wenn man wie üblich

$$\infty + \alpha = \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 12)$$

setzt.

b. Die Multiplikation der Enden. Auf der Geraden $(0, \infty)$ wählen wir einen Punkt O und errichten in O das Lot; die Enden dieses Lotes mögen mit 1 und -1 bezeichnet werden. Ein Ende werde *positiv* genannt, wenn es auf derselben Seite der Geraden $(0, \infty)$ liegt wie das Ende 1 ; ein Ende werde *negativ* genannt, wenn es auf derselben Seite der Geraden $(0, \infty)$ liegt wie das Ende -1 . Wie wir schon gesehen haben, (Satz 6) gibt es durch irgend ein von Null verschiedenes Ende ξ aus \mathfrak{h} eine Senkrechte auf der Geraden $(0, \infty)$; das andere Ende dieser Senkrechten ist dann $-\xi$. Die Spiegelung an der Geraden $(-\xi, \xi)$ werden wir \mathfrak{P}_ξ nennen; offenbar kommen \mathfrak{P}_ξ und $\mathfrak{P}_{-\xi}$ auf dasselbe hinaus.

Für die Enden von \mathfrak{h} können wir folgendermassen eine zweite Verknüpfung erklären. Es seien α und β irgend zwei von Null verschiedene Elemente von \mathfrak{h} . Auf Grund des Satzes 10 gibt es eine eindeutig bestimmte Gerade $(\pi, -\pi)$ derart, dass für die Spiegelung \mathfrak{P}_π an dieser Geraden die Beziehung

$$\mathfrak{P}_\pi = \mathfrak{P}_\beta \mathfrak{P}_1 \mathfrak{P}_\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 13)$$

erfüllt ist. Wir werden das positive bzw. das negative Ende der Geraden $(\pi, -\pi)$ das *Produkt* $\alpha\beta$ der Enden α und β nennen, je nachdem die Enden α und β entweder beide positiv bzw. beide negativ, oder eines positiv und das andere negativ ist. Wir ergänzen diese Definition mit der Festsetzung:

$$\alpha \cdot 0 = 0 \cdot \alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 14)$$

für jedes Element α von \mathfrak{h} .

19. Es gilt das kommutative Gesetz:

$$\alpha\beta = \beta\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 15)$$

Für von Null verschiedene Elemente beweist man die Behauptung wie Satz 12. Natürlich musz man jetzt auch noch auf die Vorzeichen achten.

20. Es gilt das assoziative Gesetz:

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1, 16)$$

Für von Null verschiedene Elemente wie Satz 13.

Wir sind jetzt imstande den Beweis des folgenden Satzes zu geben:

27. Die Multiplikation ist distributiv in bezug auf die Addition:

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.23)$$

Dem Beweis dieser Behauptung schicken wir folgende Bemerkung voraus. Es sei \mathfrak{B} irgend eine Bewegung und \mathfrak{S} eine Spiegelung in bezug auf die Gerade g . Dann ist $\mathfrak{B}\mathfrak{S}\mathfrak{B}^{-1}$ die Spiegelung in bezug auf die Gerade, welche durch die Bewegung \mathfrak{B} aus der Geraden g hervorgeht. Dabei ist \mathfrak{B}^{-1} wie üblich die inverse Bewegung von \mathfrak{B} .

Es sei nun α ein positives Element aus \mathfrak{h} , und β und γ irgend zwei Elemente aus \mathfrak{h} . Da die Bewegungen $\mathfrak{P}_{\sqrt{\alpha}}$, \mathfrak{P}_1 und $\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}}$ einander invers sind, gilt offenbar auf Grund der soeben gemachten Bemerkung:

$$\mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1\mathfrak{S}_\beta\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}} = \mathfrak{S}_{\alpha\beta} \quad . \quad . \quad . \quad . \quad . \quad (1.24)$$

Wenden wir nun die Formel (1.24) an auf die Beziehung:

$$\mathfrak{S}_{\beta+\gamma} = \mathfrak{S}_\gamma\mathfrak{S}_0\mathfrak{S}_\beta$$

dann finden wir:

$$\begin{aligned} \mathfrak{S}_{\alpha(\beta+\gamma)} &= \mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1\mathfrak{S}_{\beta+\gamma}\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}} \\ &= \mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1\mathfrak{S}_\gamma\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}} \cdot \mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1\mathfrak{S}_0\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}} \cdot \mathfrak{P}_{\sqrt{\alpha}}\mathfrak{P}_1\mathfrak{S}_\beta\mathfrak{P}_1\mathfrak{P}_{\sqrt{\alpha}} \\ &= \mathfrak{S}_{\alpha\gamma}\mathfrak{S}_0\mathfrak{S}_{\alpha\beta} = \mathfrak{S}_{\alpha\beta+\alpha\gamma}. \end{aligned}$$

Damit ist die Richtigkeit des Satzes für ein positives Element α erwiesen. Für $\alpha = 0$ ist die Behauptung trivial und für ein negatives Element α sieht man die Richtigkeit des Satzes ohne Mühe durch formelles Rechnen ein.

Wir können folgendermassen zusammenfassen:

28. Die Elemente von \mathfrak{h} bilden gegenüber den oben erklärten Verknüpfungen einen Körper.

Denn für \mathfrak{h} sind ja die Körperpostulate erfüllt.

Wegen des Satzes 26 gilt:

29. Im Körper \mathfrak{h} ist jede quadratische Gleichung mit nicht-negativer Diskriminante lösbar.

Weiter haben wir:

30. Der Körper \mathfrak{h} ist angeordnet.

Für die Elemente von \mathfrak{h} ist nämlich die Eigenschaft, positiv zu sein, definiert und für jedes Element α von \mathfrak{h} gilt genau eine der Aussagen: α ist positiv, α ist Null, $-\alpha$ ist positiv. Dabei kommt es auf dasselbe hinaus, wenn man sagt: α ist positiv, oder $-\alpha$ ist negativ. Weiter sind mit α und β auch $\alpha + \beta$ und $\alpha\beta$ positiv. Für das Produkt leuchtet dies sofort ein auf Grund der Definition der Multiplikation. Für die Summe kann man die Richtigkeit der Behauptung folgendermassen einsehen. Es sei P irgend ein Punkt auf der Geraden $(0, \infty)$ und A bzw. B das Bild dieses Punktes bei der Bewegung \mathfrak{S}_α bzw. \mathfrak{S}_β . Die Bewegung $\mathfrak{S}_\beta\mathfrak{S}_0\mathfrak{S}_\alpha$ führt offenbar A in B über und folglich liegen A und B symmetrisch in bezug auf die Gerade $(\alpha + \beta, \infty)$. Das bedeutet aber, dass diese Gerade ebenso wie die Punkte A und B auf derselben Seite der Geraden $(0, \infty)$ liegen wie das Ende 1, und das wollten wir zeigen. Damit ist aber auch der Beweis des Satzes schon erbracht.

Wir können nun in bekannter Weise die Beziehungen „größer als“ und „kleiner als“ einführen und dafür die üblichen Eigenschaften herleiten.

Zum Schluss bemerken wir noch, dass auch das Ende ∞ in die Rechnungen hineingezogen werden kann, wenn man die üblichen Verabredungen trifft.

(To be continued.)

Mathematics. — On the affirmative content of PEANO's theorem on differential equations. By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 28, 1942.)

1. **An example.** — The right member of the differential equation

$$\frac{dx}{dt} = 3\mu(x)^{\frac{1}{3}}, \quad \mu(x) = \text{Max}(x, 0),$$

is defined¹⁾ and continuous for every real x . The solution passing through $t=0$, $x=a$, a being a given real number, is

$$\begin{aligned} x &= \text{Max}((t+a)^3, 0) && \text{if } a > 0, \\ x &= \text{Max}((t-b)^3, 0) && \text{if } a = 0 \quad (b \geq 0 \text{ arbitrary}), \\ x &= a && \text{if } a < 0. \end{aligned}$$

If, however, neither $a > 0$, nor $a = 0$, nor $a < 0$ can be ascertained²⁾, and if t is any positive number, then the value of x can not be localised closer than within an interval containing $\langle 0, t \rangle$. In particular an arbitrary close approximation of x is not possible for any $t > 0$.

PEANO's famous theorem³⁾ states the existence of at least one solution, passing through a given point $(t_0, x_{0\mu})$, of the differential equations $dx_i/dt = f_i(t, x_\mu)$, provided $f_i(t, x_\mu)$ is continuous in a neighbourhood of $(t_0, x_{0\mu})$. Its demonstration^{3a)} uses a repeated application of BOLZANO-WEIERSTRASS' theorem, which is known⁴⁾ not to correspond with a generally explicitly achievable construction, and therefore is not recognised by the intuitionists. The example shows that the same holds true, not only for the proof of PEANO's theorem, but also for the theorem itself. If more stringent conditions (e.g. of LIPSCHITZ' type, or even somewhat weaker ones) are imposed on $f_i(t, x_\mu)$, it can be shown that the ordinary methods of CAUCHY-LIPSCHITZ or of CAUCHY-PICARD can be brought into a constructive form. If $f_i(t, x_\mu)$ is only continuous, our example shows this to be impossible. In that case we can only try to construct the whole set of solutions, passing through a given point, at once.

¹⁾ The maximum m of two real numbers x and y is a well-defined real number, even though it may not be possible to prove either $m = x$ or $m = y$. Cf ²⁾.

²⁾ The existence of such numbers, which was first proved by L. E. J. BROUWER (Cf. e.g. *Begründung der Mengenlehre I*, Verh. Koninkl. Akad. v. Wet., Amsterdam, XII; *Wis- en Natuurk. Tijdschr.* 2, 1923; *Monatsh. f. Math. en Physik* 36, 1929) is well known to-day.

³⁾ G. PEANO, *Démonstration de l'intégrabilité des équations différentielles ordinaires*. *Math. Ann.* 37, 182—228. 1890. Cf. also G. MIE, id. 43, 553—568. 1893. The proof is reproduced e.g. by C. CARATHEODORY, *Variationsrechnung*, and E. KAMKE, *Differentialgleichungen reeller Funktionen*. Our present demonstration is much more like PEANO's original one, which uses CANTOR's instead of BOLZANO-WEIERSTRASS' theorem. Cf. D. VAN DANTZIG, A remark and a problem concerning the intuitionistic form of CANTOR's intersection theorem, these *Proceedings*, 45, 374—375, 1942¹⁰⁾.

^{3a)} As it is simplified by C. ARZELÁ, *Bologna Mem.* (5) 5, 257—270, 1895; (5) 6, 131—140, 1896, P. MONTEL, *Ann. Ec. Norm.* 24, 264—283, 1907.

⁴⁾ Cf. e.g. L. E. J. BROUWER, l.c. ²⁾.

2. **Difference inequalities.** — We consider the system of r differential equations

$$\frac{dx_\lambda}{dt} = f_\lambda(t, x_\mu) \quad (\lambda, \mu = 1, \dots, r), \quad (1)$$

where $|t - t_0| \leq a$, $|x_\mu - x_{0\mu}| \leq b_\mu$, whereas for $(t_1, x_{1\mu})$, $(t_2, x_{2\mu})$ lying in this range, $|f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq \varepsilon_\lambda$ if $|t_1 - t_2| \leq \delta(\varepsilon_\lambda)$, $|x_{1\mu} - x_{2\mu}| \leq \delta_\mu(\varepsilon_\lambda)$. Then the functions f_λ are bounded: $|f_\lambda(t, x_\mu)| \leq N_\lambda$. Putting $a' = \text{Min}(a, N' b_1/N_1, \dots, N' b_r/N_r)$, $t' = (t - t_0)/a'$, $x'_\mu = N'(x_\mu - x_{0\mu})/a' N_\mu$, $f'_\lambda(t', x'_\mu) = N' f_\lambda(t, x_\mu)/N_\lambda$, $\delta'(\varepsilon) = \text{Min}(\delta(\varepsilon N_\lambda/N')/a')$, $N' \delta_\mu(\varepsilon N_\lambda/N')/a' N_\mu$, where $0 < N' < 1$, e.g. $N' = \frac{3}{4}$, we find that, dropping the accents again, the equations (1) are invariant. The range becomes $|t| \leq 1$, $|x_\mu| \leq 1$. In this range $|f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq \varepsilon$ if $|t_1 - t_2| \leq \delta(\varepsilon)$, $|x_{1\mu} - x_{2\mu}| \leq \delta(\varepsilon)$, and $|f_\lambda(t, x_\mu)| \leq N = \frac{3}{4} < 1$.

Let further $l(k)$ be natural numbers with $l = l(k) \geq \text{Max}(-2 \log \delta(2^{-k} \beta), l(k-1))$, where $0 < \beta < \text{Min}(\frac{1}{16}, 1 - N)$, e.g. $\beta = \frac{1}{16}$, so that $\beta + N = \frac{13}{16} < 1$.

Then

$$\left. \begin{array}{l} |t_1 - t_2| \leq 2^{-l} \\ |x_{1\mu} - x_{2\mu}| \leq 2^{-l} \end{array} \right\} \rightarrow |f_\lambda(t_1, x_{1\mu}) - f_\lambda(t_2, x_{2\mu})| \leq 2^{-k} \beta. \quad (2)$$

If T, X_μ are integers with $|T| \leq 2^l$, $|X_\mu| \leq 2^{k+l}$ ($l = l(k)$), then integers $F_{k\lambda}(T, X_\mu)$ exist, such that

$$|F_{k\lambda}(T, X_\mu) - 2^k f_\lambda(2^{-l} T, 2^{-k-l} X_\mu)| \leq \alpha, \quad (3)$$

where $\frac{1}{2} < \alpha < \frac{2}{3} - \beta$, e.g. $\alpha = \frac{9}{16}$, so that $3(\alpha + \beta) = \frac{15}{8} < 2$. From the inequalities (3) follows $|F_{k\lambda}(T, X_\mu)| \leq 2^k$, the left member being an integer $\leq \alpha + 2^k N < 1 + 2^k$.

Moreover

$$|F_{k+1,\lambda}(2^{l(k+1)-l(k)} T, 2^{l(k+1)-l(k)+1} X_\mu) - 2 F_{k\lambda}(T, X_\mu)| \leq 1, \quad (4)$$

$$\left. \begin{array}{l} |T - T'| \leq 2 \\ |X_\mu - X'_\mu| \leq 2^{k+1} \end{array} \right\} \rightarrow |F_{k\lambda}(T, X_\mu) - F_{k\lambda}(T', X'_\mu)| \leq 1, \quad (5)$$

the left members being integers $\leq 2(\alpha + \beta) < 2$. More generally we have with $l' = l(k')$:

$$\left. \begin{array}{l} |2^{-l} T - 2^{-l'} T'| \leq 2^{-l} + 2^{-l'} \\ |2^{-k-l} X_\mu - 2^{-k'-l'} X'_\mu| \leq 2^{-l} + 2^{-l'} \end{array} \right\} \rightarrow |2^{-k} F_{k\lambda}(T, X_\mu) - 2^{-k'} F_{k\lambda}(T', X'_\mu)| \leq (\alpha + \beta)(2^{-k} + 2^{-k'}), \quad (6)$$

as is seen by twice applying (2) and (3) with $t_1 = 2^{-l} T$, $x_{1\mu} = 2^{-k-l} X_\mu$ and $t_1 = 2^{-l'} T'$, $x_{1\mu} = 2^{-k'-l'} X'_\mu$ respectively, and $t_2 = (T + T') / (2^l + 2^{l'})$, $x_{2\mu} = (2^{-k} X_\mu + 2^{-k'} X'_\mu) / (2^l + 2^{l'})$.

The system of differential equations (1) is now replaced by the sequence of systems of inequalities in finite differences

$$|\Delta X_{k\lambda}(T) - F_{k\lambda}(T, X_{k\mu}(T))| \leq 1, \quad (7)$$

where $0 \leq T \leq 2^l - 1$, $l = l(k)$, $\Delta X_{k\lambda}(T) = X_{k\lambda}(T+1) - X_{k\lambda}(T)$. For each fixed value of the integer $k \geq 0$ they have a finite number of solutions in integers $X_{k\lambda}(T)$, among which at least one satisfies the initial condition $X_{k\lambda}(0) = 0$; it is determined by

$X_{k\lambda}(T+1) = X_{k\lambda}(T) + F_{k\lambda}(T, X_{k\lambda}(T))$. The solutions correspond with CAUCHY-polygons, having their edges at $t = 2^{-l}T$, $x_\lambda = 2^{-k-l}X_{k\lambda}(T)$, $0 \leq T \leq 2^l$.

3. **Top-functions.** We define:

$$l_0 = 0, \lambda_0 = 1, l_n = [2 \log n]^5, \lambda_n = 2^{-l_n-1} \quad (n \geq 1), \quad . \quad . \quad (8)$$

$$r_{-1} = 0, r_0 = 1, r_n = (2n+1)\lambda_n - 1, (n \geq 1), \quad . \quad . \quad (9)$$

Further, I_n for $n \geq 1$ is the closed interval $I_n = \langle r_n - \lambda_n, r_n + \lambda_n \rangle$. In particular $I_1 = \langle 0, 1 \rangle$. An interval I_n is contained in I_m if and only if integers $p \geq 0$ and q exist, $0 \leq q \leq 2^p - 1$, such that

$$n = 2^p m + q, \quad \text{i.e.} \quad m = [2^{-p}n], \quad . \quad . \quad . \quad (10)$$

In this case $l_n = l_m + p$, $\lambda_n = 2^{-p}\lambda_m$,

$$r_n - r_m = \{2^{-p}(2q+1) - 1\}\lambda_m = (2q+1)\lambda_n - \lambda_m \quad . \quad . \quad (11)$$

The well-known "top-functions" $u_n(t)$ ($0 \leq t \leq 1$, $n \geq -1$) are defined by

$$u_{-1}(t) = 1, \quad u_n(t) = \text{Max}(0, 1 - |r_n - t| \lambda_n^{-1}) \quad (n \geq 0), \quad . \quad (12)$$

In particular $u_m(r_n) = 0$, unless n has the form (10), e.g. if $m > n$. If

$$\varphi(t) = \sum_{-1}^{\infty} c_n u_n(t) \quad . \quad . \quad . \quad (13)$$

is uniformly convergent for $0 \leq t \leq 1$, then $c_{-1} = \varphi(0)$, $c_0 = \varphi(1) - \varphi(0)$,

$$c_n = \varphi(r_n) - \frac{1}{2} \{ \varphi(r_n + \lambda_n) - \varphi(r_n - \lambda_n) \} \quad (n \geq 1), \quad . \quad . \quad (14)$$

Hence the c_n are uniquely determined by $\varphi(t)$. If necessary we write $c_n[\varphi]$ instead of c_n . If and only if $c_n = 0$ for $n \geq 2^l$, the graph of $\varphi(t)$ is a polygon with its edges on $t = 2^{-l}j$, $0 \leq j \leq 2^l$.

If $|\varphi(t)| \leq M$ for every $t \in I_1$, then by (14) $|c_n[\varphi]| \leq 2M$ for every $n \geq -1$. If therefore $|\varphi(t) - \psi(t)| \leq \varepsilon$ ($0 \leq t \leq 1$), then $|c_n[\varphi] - c_n[\psi]| = |c_n[\varphi - \psi]| \leq 2\varepsilon$ ($n \geq -1$). Hence, if $\varphi(t) = \lim \varphi_\nu(t)$ uniformly in t , then $c_n[\varphi] = \lim c_n[\varphi_\nu]$ uniformly in n . If the variation of φ is bounded on I_1 :

$$\left| \frac{\Delta \varphi}{\Delta t} \right| = \left| \frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \right| \leq N \quad . \quad . \quad . \quad (15)$$

($t_1 \in I_1$, $t_2 \in I_1$, $t_1 \neq t_2$), then

$$|c_n[\varphi]| \leq N \lambda_n \quad . \quad . \quad . \quad (16)$$

for every $n \geq 0$. In fact, for $n=0$ (16) is trivial, and for $n \geq 1$ we have

$$|c_n| \leq \frac{1}{2} |\varphi(r_n) - \varphi(r_n - \lambda_n)| + \frac{1}{2} |\varphi(r_n) - \varphi(r_n + \lambda_n)| \leq N \lambda_n.$$

Hence in this case $\sum_{-1}^{\infty} |c_n|^2$ is uniformly convergent, as for $k \geq 1$

$$\sum_{2^k}^{\infty} |c_n|^2 = \sum_k^{\infty} \sum_{j=0}^{2^l-1} |c_{2^l+j}|^2 \leq \sum_k^{\infty} 2^l (2^{l-1} N)^2 = 2^{-k-1} N^2.$$

5) Though the entier-function $[x]$ can not be determined for every real x , it can, e.g. if x is a rational number or its logarithm, upon which it will only be applied here.

Hence the coefficients of the development of a function of bounded variation are the coordinates of a point of HILBERT space, belonging to the so-called "compact quadra"

$$\sum_0^{\infty} |c_n|^2 \leq \frac{2}{3} N^2.$$

If, at the other hand, (16) holds, then for $0 \leq j \leq 3 \cdot 2^p - 1$

$$\left| \sum_{2^p+j}^{2^{p+2}-1} c_n [\varphi] u_n(t) \right| \leq 2^{-p-1} N \sum_{2^p}^{2^{p+1}-1} \{u_n(t) + \frac{1}{2} u_{2n}(t) + \frac{1}{2} u_{2n+1}(t)\} \leq 2^{-p-1} N,$$

as for every t at most one of the terms between the curved brackets can be > 0 and then remains ≤ 1 . Hence

$$\left| \sum_k^{\infty} c_n [\varphi] u_n(t) \right| \leq \sum_q^{\infty} 2^{-l_k-2q-1} = \frac{4}{3} N \lambda_k,$$

so that (13) converges uniformly and then represents a uniformly continuous function $\varphi(t)$ with coefficients (14). In particular, in this case

$$|\varphi(t)| \leq |c_{-1}[\varphi]| + \frac{4}{3} N. \quad . \quad . \quad . \quad . \quad (17)$$

The variation of φ , however, need not be bounded, as is seen from the example $c_n = \lambda_n$ ($n \geq 0$), where $\{\varphi(2^{-l}) - \varphi(0)\} 2^l = l + 1$.

The direction coefficients

$$m_n = m_n[\varphi] = (\Delta \varphi / \Delta t)_{I_n} = \{\varphi(r_n + \lambda_n) - \varphi(r_n - \lambda_n)\} / 2 \lambda_n \quad . \quad (18)$$

can be expressed by the coefficients c_n :

$$m_n = \sum_0^{l_n} (-1)^{[2^{-j} j_n]} c_{[2^{-j-1} n]} \lambda_{[2^{-j-1} n]}^{-1} \quad (n \geq 1) \quad . \quad . \quad . \quad (19)$$

where $j_n = n - 2^{l_n}$; $\lambda_{[2^{-j-1} n]}^{-1} = 2^{l_n-j}$. In fact, for $n=1$ (19) states that $m_1 = c_0$, which is trivial. If (19) holds for a certain value of n , then $r_{2n+q} = r_n + (q - \frac{1}{2}) \lambda_n$, $0 \leq q \leq 1$ (Cf. (11) with $p=1$), and $m_{2n+q} = m_n + (-1)^q c_n \lambda_n^{-1}$. As $j_{2n+q} = 2j_n + q$ and $[2^{-i} q] = 0$ for $i \geq 1$, (19) is found to hold for $2n+q$ instead of n . Hence it holds for every n .

At the other hand, $c_n = \frac{1}{2} \lambda_n (m_{2n} - m_{2n+1})$, $m_n = \frac{1}{2} (m_{2n} + m_{2n+1})$. Hence, generally for every $g \geq 1$, $n \geq 1$:

$$m_n = 2^{-g} \sum_0^{2^g-1} m_{2^g n+j}, \quad . \quad . \quad . \quad . \quad (20)$$

$$c_n = 2^{-g} \lambda_n \sum_0^{2^g-1} (-1)^{[2^{-g+1} j]} m_{2^g n+j}, \quad . \quad . \quad . \quad (21)$$

By (21) we can determine the c_n for $n < 2^l$ if the m_n are known for $2^l \leq n < 2^{l+1}$. For, taking $n = 2^{l-g} + h$, $1 \leq g \leq l$, $0 \leq h \leq 2^{l-g} - 1$, (21) becomes

$$c_{2^{l-g}+h} = 2^{-l-1} \sum_0^{2^g-1} (-1)^{[2^{-g+1} j]} m_{2^l+2^g h+j}, \quad c_0 = 2^{-l} \sum_0^{2^l-1} m_{2^l+j}. \quad (22)$$

In particular (22) determines all coefficients of a polygonal line, having its edges on $t = 2^{-l}j$ ($0 \leq j \leq 2^l$), since the coefficients c_n with $n \geq 2^l$ then vanish.

Finally we conclude from (21) that a variation $\leq \varepsilon$ of the m_n ($n \geq 1$) leads for each $k \geq 0$ to a variation $\leq \varepsilon \lambda_k$ of c_k . At the other hand, we can by (19) only conclude from a variation $\leq \varepsilon \lambda_k$ of the c_k ($k \geq 0$) to a variation $\leq (l_n + 1)\varepsilon$ of m_n .

4. **The theorem.**⁶⁾ — Let S'_k be the set of all systems of real numbers $\gamma_{n\lambda}$, such that $\gamma_{-1,\lambda} = 0$ and that for each $n \geq 1$ and for at least one $\varphi_\lambda(t)$ corresponding with a solution $X_{k\lambda}(T)$ with $X_{k\lambda}(0) = 0$ of (7)

$$\left| \sum_{j=0}^{l_n} (-1)^{[2^{-j}j n]} 2^{-j+l_n} \gamma_{[2^{-j-1}n],\lambda} - m_n [\varphi_\lambda] \right| \leq 2^{-k+1} \quad . \quad . \quad (23)$$

S'_k contains at least one element, viz $\gamma_{n\lambda} = c_n [\varphi_\lambda]$, where φ_λ corresponds with a solution of (7) which certainly exists, as was said before. Further let k_h be natural numbers with $k_{h+1} \geq \text{Max}(k_h + f, l(k_h + 1) + 2)$, where $f \geq 3 - 2 \log \beta$, e.g. $f = 7$, and $S_h = S'_{k_h}$. Then S'_k and S_h each consist of a finite number of „compact“ paralleloptopes in HILBERT space. We prove now:

A. $S_{h+1} \subset S_h$.

B. If a solution $x_\lambda = \psi_\lambda(t)$ of (1) with $\psi_\lambda(0) = 0$ exists, its sequence of coefficients $c_n [\psi_\lambda]$ belongs to every S_h .

C. If $\gamma_{n\lambda}$ belongs to every S_h , they are the coefficients of a solution of the differential equations (1), satisfying $\psi_\lambda(0) = 0$.

This will be proved by means of two lemma's.

Lemma 1. If the functions $\psi_\lambda(t)$ are continuous for $0 \leq t \leq 1$, $|\psi_\lambda(t)| \leq 1$, $\psi_\lambda(0) = 0$, and

$$\left| [\psi_\lambda(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t)) dt \right| \leq 2^{-k} \beta |t_2 - t_1| \quad . \quad . \quad . \quad (24)$$

for arbitrary t_1, t_2 in I_1 , and $k = k_h$, then the coefficients $c_n [\psi_\lambda]$ belong to S_h .

Lemma 2. If the real numbers $\gamma_{n\lambda}$ belong to S_{h+1} , they are the coefficients $c_n [\psi_\lambda]$ of continuous functions $\psi_\lambda(t)$ with $|\psi_\lambda(t)| \leq 1$, $\psi_\lambda(0) = 0$, and satisfying the inequalities (24).

It is trivial that the two lemma's imply statement A. Further, $\psi_\lambda(t)$ satisfying (1) is equivalent with the identity $[\psi_\lambda(t)]_{t_1}^{t_2} = \int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t)) dt$, for all t_1, t_2 in I_1 hence with validity of the integral inequalities (24) for every k . This leads immediately to statements B and C.

5. **Proof of Lemma 1.** — Let $\psi_\lambda(t)$ be continuous with $\psi_\lambda(0) = 0$, $|\psi_\lambda(t)| \leq 1$ and satisfy (24) for $k = k_h$. Then integers $X_{k\lambda}(T)$ ($0 \leq T \leq 2^l - 1$) exist, satisfying $X_{k\lambda}(0) = 0$ and

$$|X_{k\lambda}(T) - 2^{k+l} \psi_\lambda(2^{-l} T)| \leq \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

⁶⁾ The present result was found in 1939; a few simplifications date from 1940. I express my gratitude to my then assistants Mr. J. C. BOLAND and Mr. J. DE IONGH, who have checked some of the calculations and improved the demonstration by useful remarks. Mr. DE IONGH moreover contributed the formulae (19), (21) as a substitute for two somewhat lengthier formulae of the same type.

Then $|X_{k\lambda}(T)| \leq 2^{k+l}$, and we obtain successively for $2^{-l}T \leq t_1 \leq t \leq t_2 \leq 2^{-l}(T+1)$:

$$1^\circ. \quad \left| \int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t)) dt \right| \leq N |t_2 - t_1|.$$

$$2^\circ. \quad |\psi_\lambda(t) - \psi_\lambda(2^{-l}T)| \leq 2^{-k} \beta (t - 2^{-l}T) + \left| \int_{2^{-l}T}^t f_\lambda(t, \psi_\mu(t)) dt \right| \leq (2^{-k} \beta + N)(t - 2^{-l}T) < 2^{-l}.$$

$$3^\circ. \quad |2^k f_\lambda(t, \psi_\mu(t)) - F_{k\lambda}(T, X_{k\mu}(T))| \leq \alpha + \beta \quad \text{by } (2), (3).$$

$$4^\circ. \quad |[\psi_\lambda(t)]_{t_1}^{t_2} - 2^{-k} F_{k\lambda}(T, X_{k\mu}(T))(t_2 - t_1)| \leq 2^{-k} \beta |t_2 - t_1| + \int_{t_1}^{t_2} |f_\lambda(t, \psi_\mu(t)) - 2^{-k} F_{k\lambda}(T, X_{k\mu}(T))| dt \leq 2^{-k}(\alpha + 2\beta) |t_2 - t_1|.$$

in particular $|2^{k+l} \{ \psi_\lambda(2^{-l}(T+1)) - \psi_\lambda(2^{-l}T) \} - F_{k\lambda}(T, X_{k\mu}(T))| \leq \alpha + 2\beta$.

$$5^\circ. \quad |\triangle X_{k\lambda}(T) - F_{k\lambda}(T, X_{k\mu}(T))| \leq 1$$

the left members being integers $\leq 3\alpha + 2\beta < 2$. Hence the integers $X_{k\lambda}(T)$ satisfy the difference inequalities (7). If $\varphi_\lambda(t)$ are the corresponding CAUCHY polygons, $m_{2^l+T}[\varphi_\lambda] = 2^{-k} \Delta X_{k\lambda}(T)$ being their direction coefficients, we obtain:

$$6^\circ. \quad |2^k f_\lambda(t, \psi_\mu(t)) - \triangle X_{k\lambda}(T)| \leq 1 + \alpha + \beta \quad \text{by } 3^\circ, 5^\circ.$$

Hence for $t_1 \neq t_2$:

$$\begin{aligned} 7^\circ. \quad \left| \frac{\triangle \psi_\lambda}{\triangle t} - \frac{\triangle \varphi_\lambda}{\triangle t} \right| &= \left| \frac{[\psi_\lambda(t)]_{t_1}^{t_2}}{t_2 - t_1} - 2^{-k} \triangle X_{k\lambda}(T) \right| \leq \\ &\leq \frac{1}{|t_2 - t_1|} \int_{t_1}^{t_2} |f_\lambda(t, \psi_\mu(t)) - 2^{-k} \triangle X_{k\lambda}(T)| dt + 2^{-k} \beta \leq \\ &\leq 2^{-k} (1 + \alpha + 2\beta) < 2^{-k+1}. \end{aligned}$$

Hence $|m_n[\psi_\lambda] - m_n[\varphi_\lambda]| \leq 2^{-k+1}$ for each $n \geq 2^l$ and then by (20) for $n \geq 1$. It follows then from (19) that $\gamma_{n\lambda} = c_n[\psi_\lambda]$ satisfies (23), hence belongs to $S_{h'}$, which proves the lemma.

6. Proof of Lemma 2. — Let $\gamma_{n\lambda}$ belong to S_{h+1} . Then CAUCHY polygons $\varphi_\lambda(t)$ with edges $(2^{-l'}T, X_{k'\lambda}(t))$, $0 \leq T \leq 2^{l'}$, $k' = k_{h+1}$, $l' = l(k')$ exist, satisfying (23) with k' instead of k , for $n \geq 2^{l'}$ and then by (20) also for $n \geq 1$. Hence by (21): $|\gamma_{n\lambda} - c_n[\varphi_\lambda]| \leq 2^{-k'+1} \lambda_n$. Hence by an argument like before, $\psi_\lambda(t) = \sum_0^\infty \gamma_{n\lambda} u_n(t)$ converges uniformly and by (17)

$$|\psi_\lambda(t) - \varphi_\lambda(t)| = \left| \sum_0^\infty (\gamma_{n\lambda} - c_n[\varphi_\lambda]) u_n(t) \right| \leq \frac{1}{3} \cdot 2^{-k'+3} < 2^{-l(k+1)}, \quad (26)$$

as $k' \geq l(k+1) + 2$.

Hence, with $2^{-l'} T \leq t_1 \leq t \leq t_2 \leq 2^{-l'} (T+1)$,

$$X_{k'\lambda}(T) = 2^{k'+l'} \varphi_\lambda(2^{-l'} T), |\Delta X_{k'\lambda}(T) - F_{k'\lambda}(T, X_{k'\mu}(T))| \leq 1,$$

we have successively

1°. $|\Delta X_{k'\lambda}(T)| \leq 1 + \alpha + 2^{k'} N < (\beta + N) 2^{k'} < 2^{k'}$,
as $1 + \alpha < \frac{5}{3} < 2^{f-2} \beta < 2^{k'} \beta$. Hence $|X_{k'\lambda}(T)| \leq (1 + \alpha + 2^{k'} N) 2^{l'}$,
 $|\varphi_\lambda(t)| \leq N + 2^{-k'}(1 + \alpha)$ and $|\psi_\lambda(t)| \leq N + 2^{-k'}(\frac{1}{3} + \alpha) < N + 2^{-f+3} \leq N + \beta < 1$.

$$2^\circ. |\varphi_\lambda(t_2) - \varphi_\lambda(t_1)| = 2^{-k'} |\Delta X_{k'\lambda}(T)(t_2 - t_1)| < (\beta + N) |t_2 - t_1| \leq 2^{-l'}.$$

$$3^\circ. |\Delta X_{k'\lambda}(T) - 2^{k'} f_\lambda(t, \varphi_\mu(t))| \leq |\Delta X_{k'\lambda}(T) - F_{k'\lambda}(T, X_{k'\mu}(T))| + \\ + |F_{k'\lambda}(T, X_{k'\mu}(T)) - 2^{k'} f_\lambda(2^{-l'} T, 2^{-k'-l'} X_{k'\mu}(T))| + \\ + 2^{k'} |f_\lambda(2^{-l'} T, \varphi_\mu(2^{-l'} T)) - f_\lambda(t, \varphi_\mu(t))| \leq 1 + \alpha + \beta < 2.$$

$$4^\circ. \left| [\varphi_\lambda(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda(t, \varphi_\mu(t)) dt \right| \leq \\ \leq \int_{t_1}^{t_2} |2^{-k'} \Delta X_{k'\lambda}(T) - f_\lambda(t, \varphi_\mu(t))| dt \leq 2^{-k'+1} |t_2 - t_1|.$$

$$5^\circ. |f_\lambda(t, \varphi_\mu(t)) - f_\lambda(t, \psi_\mu(t))| \leq 2^{-k-1} \beta \text{ by (26), (2).}$$

If $t_1 = r_n - \lambda_n$, $t_2 = r_n + \lambda_n$ with $n \geq 2^{l'+1}$ we have moreover by (23):

$$6^\circ. |[\psi_\lambda(t) - \varphi_\lambda(t)]_{t_1}^{t_2}| = |t_2 - t_1| |m_n [\psi_\lambda - \varphi_\lambda]| \leq 2^{-k'+1} |t_2 - t_1|.$$

$$7^\circ. \left| [\psi_\lambda(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} f_\lambda(t, \psi_\mu(t)) dt \right| \leq |[\varphi_\lambda(t)]_{t_1}^{t_2} - \\ - \int_{t_1}^{t_2} f_\lambda(t, \varphi_\mu(t)) dt| + |[\psi_\lambda(t) - \varphi_\lambda(t)]_{t_1}^{t_2}| + \int_{t_1}^{t_2} |f_\lambda(t, \varphi_\mu(t)) - f_\lambda(t, \psi_\mu(t))| dt \leq \\ \leq (2^{-k'+1} + 2^{-k'+1} + 2^{-k-1} \beta) |t_2 - t_1| \leq 2^{-k} (2^{-f+2} + \frac{1}{2} \beta) |t_2 - t_1| \leq 2^{-k} \beta |t_2 - t_1|.$$

Hence (24) has been proved for every sufficiently small interval with dyadic rational endpoints. But both sides of (24) are the absolute values of additive interval-functions, hence (24) then holds for every interval $\subset I_1$, which proves the lemma, and therewith the theorem.

Mathematics. — *A remark and a problem concerning the intuitionistic form of CANTOR's intersection theorem.* By D. VAN DANTZIG. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 28, 1942.)

In classical mathematics the theorem proved in the preceding paper¹⁾ would imply PEANO's theorem by an application of CANTOR's theorem, which states that the intersection S of a decreasing sequence of compact non-empty subsets S_h is not empty²⁾. An analogous application concerns the maxima of a real function $f(x)$, continuous for $a \leq x \leq b$. It is easy to construct for every natural h a set S_h consisting of a finite number > 0 of closed dyadic rational intervals, such that

- A. $S_{h+1} \subset S_h$.
- B. If $f(x) \leq f(c)$ for all x with $a \leq x \leq b$, then $c \in S_h$ for every h .
- C. If $c \in S_h$ for every h , then $f(x) \leq f(c)$ for all x with $a \leq x \leq b$.

Denoting by \forall the logical "all-symbol" and by \exists the "existence-symbol"^{2a)} (in the intuitionistic sense, hence requiring the possibility of explicit arbitrarily close approximation of the "existing" entity), the "strong interpretation"³⁾ of CANTOR's theorem would require

$$\exists x \forall h, x \in S_h, \dots \dots \dots (1)$$

This, however, is certainly wrong, as in the case of the maximum of a real function is shown by a counter-example of BROUWER⁴⁾ and in the case of PEANO's theorem by the opening section of the preceding paper. At the other hand, denoting the negation-symbol by \neg , the strong interpretation of the negation of (1) would be

$$\forall x \exists h \neg, x \in S_h, \dots \dots \dots (2)$$

A counter-example with this property, however, is not possible. For it would require the constructibility of a natural number h corresponding with *any* real number in the given interval. Hence, by BROUWER's theorem⁵⁾ such a number h could be determined simultaneously for all x , i.e.

$$\exists h \forall x \neg, x \in S_h, \dots \dots \dots (3)$$

which is contradicted by the existence of at least one number x belonging to an arbitrarily given S_h .

1) D. VAN DANTZIG, On the affirmative content of PEANO's theorem on differential equations. Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 367—373 (1942).

2) "Compact sets" may be interpreted here as "gecatalogiseerd compacte soorten" as defined by BROUWER, Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 634—642, 677—678 (1927).

2a) Cf. A. HEYTING, Sber. Pr. Ak. v. Wiss. 1930, p. 42—71, 158—169. There brackets are used instead of the symbol \forall .

3) Cf. D. VAN DANTZIG, On the principles of intuitionistic and affirmative mathematics. This paper was written on bequest of the redaction of the Revista Matematica Hispano-Americana, and sent to the redaction in Febr./March 1941. Whether it has meanwhile appeared or not, could not be ascertained.

4) Cf. e.g. L. E. J. BROUWER, Wis- en Natuurk. Tijdschr. **2** (1923).

5) Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 189—193 (1924).

Hence CANTOR's theorem can only be valid in the "weak interpretation"

$$\neg \forall x \neg \forall h, x \in S_h \quad (4)$$

whereas at the other hand a counter-example could only be possible also in the weak interpretation

$$\forall x \neg \forall h \neg \neg x \in S_h \quad (5)$$

I have not succeeded, either in proving (4), nor in finding an example with (5). As it seems, the methods known to day in intuitionistic mathematics do not allow to decide between such purely negative statements like (4) and (5) ⁷⁾.

It is for this reason that I have hesitated so long to publish the preceding and the present paper. That I nevertheless have decided to publish them now has two reasons. The first of these is that if the difficulty is publicly signalled, other investigators might be induced to find general methods which allow the intuitionistic treatment of "stable statements" (i.e. statements equivalent with their double negation) ⁸⁾, obtained by "weakening up" classical statements. This would be of importance, because it is the ideal of classical mathematicians to work with stable statements only, whereas in fact classical mathematics uses a peculiar mixture of negative and affirmative statements. The other reason is that the construction of the sets S_h mentioned above, as well as the corresponding construction given in the preceding paper is purely "affirmative" ⁹⁾, i.e. does not make use of any negations (nor of unrestricted existence statements either), and is in no way influenced by the proof of a purely negative statement like (4), nor even by its refutation (5). For (5) only causes that, starting with the successive approximation of a given real number x , the relation $x \in S_h$ can not be maintained always, without an upper limit being known for the number of the step at which the process is checked; the knowledge of such an upper limit can certainly not be ascertained for every x , as this would lead to (2).

⁶⁾ As the statement $x \in S_h$ is "stable" (i.e. equivalent with its double negation), (5) can be replaced by $\forall x \neg \forall h, x \in S_h$, which is the direct negation of (4), $\forall x \Omega[x]$ always being stable, if $\Omega[x]$ is, as it is the negation of $\exists x \neg \Omega[x]$.

⁷⁾ With respect to the statements (4), (5) this was kindly confirmed to me by Dr. A. HEYTING.

⁸⁾ Cf. l.c. ³⁾.

⁹⁾ Cf. l.c. ³⁾.

¹⁰⁾ Herein lies the difference between the proofs of PEANO's theorem using CANTOR's theorem (PEANO's original proof, MIE), and those using BOLZANO-WEIERSTRASS' theorem (MONTEL, PERRON, a.o.): Though it is not possible in general to construct a point of the intersection, the decreasing sequence of compact sets itself, of which CANTOR's theorem speaks, can be constructed. It is, however, in general impossible to construct not only the *limit* of a convergent subsequence, the existence of which is stated by BOLZANO-WEIERSTRASS, but even such a subsequence itself.

Mathematics. — *Over reeksen en bepaalde integralen, waarbij functies van BESSEL optreden.* I. Door J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of March 28, 1942.)

Reeds eerder hebben we de volgende formule afgeleid ¹⁾:

$$\int_0^x I_\nu(x-a) I_\varrho(a) (x-a)^{\nu} a^{\varrho} da = \frac{\Gamma(\nu + \frac{1}{2}) \Gamma(\varrho + \frac{1}{2})}{\Gamma(\nu + \varrho + 1) \sqrt{\pi}} x^{\nu+\varrho} I_{\nu+\varrho+\frac{1}{2}}(x), \quad (1)$$

waarin ν en ϱ willekeurige getallen voorstellen, waarvan de reële gedeelten $> -\frac{1}{2}$ zijn, hetgeen we aldus aangeven: $R(\nu) > -\frac{1}{2}$, $R(\varrho) > -\frac{1}{2}$.

Met behulp van een bekende absoluut convergente reeks kunnen we van (1) verschillende toepassingen maken, die tot belangrijke resultaten aanleiding geven.

1. Beschouwen we n.l. de reeksontwikkeling ²⁾:

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu - \nu + n)}{n! \Gamma(\mu + n + 1)} \left(\frac{x}{2}\right)^{\nu+n} I_{\nu+n}(x) = \frac{\Gamma(\mu - \nu)}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^{2\nu-\mu} I_{\mu}(x), \quad (I)$$

μ en ν willekeurig.

We leiden hieruit de bijzondere af voor $\mu = \nu - \frac{1}{2}$:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\nu + n + \frac{1}{2})} \left(\frac{x}{2}\right)^{\nu+n} I_{\nu+n}(x) = \frac{\Gamma(-\frac{1}{2})}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^{\nu+\frac{1}{2}} I_{\nu-\frac{1}{2}}(x), \quad (II)$$

ν willekeurig.

Door nu in (1) ν te vervangen door $\nu + n$ en daarna beide leden te vermenigvuldigen met $\frac{\Gamma(n - \frac{1}{2})}{2^{\nu+n} n! \Gamma(\nu + n + \frac{1}{2})}$ volgt, na sommatie over n van 0 tot ∞ , onder toepassing van II in het linker lid onder het integraalteeken en evenzoo in het rechter lid, na vooraf in II ν door $\nu + \varrho + \frac{1}{2}$ te hebben vervangen:

$$\int_0^x I_{\nu-\frac{1}{2}}(x-a) I_{\varrho}(a) (x-a)^{\nu+\frac{1}{2}} a^{\varrho} da = \frac{\Gamma(\nu+1) \Gamma(\varrho + \frac{1}{2})}{\Gamma(\nu + \varrho + \frac{3}{2}) \sqrt{\pi}} \left(\frac{x}{2}\right)^{\nu+\varrho+\frac{1}{2}} I_{\nu+\varrho}(x), \quad (2)$$

$$R(\nu) > -\frac{1}{2}, R(\varrho) > -\frac{1}{2}.$$

Deze formule hebben we reeds eerder langs anderen weg afgeleid ³⁾.

Op grond van de betrekkingen:

$$I_{-\frac{1}{2}}(y) = \sqrt{\frac{2}{\pi y}} \cos y \quad \text{en} \quad I_{\frac{1}{2}}(y) = \sqrt{\frac{2}{\pi y}} \sin y \quad . \quad . \quad . \quad (a)$$

¹⁾ Nieuw Archief voor Wiskunde (2) VII, 1907, p. 400, (33).

²⁾ NIELSEN, Handbuch der Theorie der Cylinderfunktionen 1904, p. 268, (1).

³⁾ Proc. Kon. Akad. v. Wetensch., Amsterdam, **34**, N^o. 1, 149, (2) (1931).

volgen uit (1) voor $\nu = \frac{1}{2}$ en uit (2) voor $\nu = 0$:

$$\int_0^x I_\varrho(a) a^\varrho \sin(x-a) da = \frac{x^{\varrho+1}}{2\varrho+1} I_{\varrho+1}(x), \quad R(\varrho) > -\frac{1}{2} \quad (3)$$

$$\int_0^x I_\varrho(a) a^\varrho \cos(x-a) da = \frac{x^{\varrho+1}}{2\varrho+1} I_\varrho(x), \quad R(\varrho) > -\frac{1}{2} \quad (4)$$

Vermenigvuldigen we beide leden van (4) met $\sin x$ resp. $\cos x$ en die van (3) met $\cos x$ resp. $\sin x$, dan vinden we na aftrekking en optelling der overeenkomstige leden:

$$\int_0^x I_\varrho(a) a^\varrho \sin a da = \frac{x^{\varrho+1}}{2\varrho+1} \{ \sin x I_\varrho(x) - \cos x I_{\varrho+1}(x) \}, \quad R(\varrho) > -\frac{1}{2} \quad (5)$$

$$\int_0^x I_\varrho(a) a^\varrho \cos a da = \frac{x^{\varrho+1}}{2\varrho+1} \{ \cos x I_\varrho(x) + \sin x I_{\varrho+1}(x) \}, \quad R(\varrho) > -\frac{1}{2} \quad (6)$$

$$\int_0^x I_\varrho(a) a^\varrho \sin(2x-a) da = \frac{x^{\varrho+1}}{2\varrho+1} \{ \sin x I_\varrho(x) + \cos x I_{\varrho+1}(x) \}, \quad R(\varrho) > -\frac{1}{2} \quad (7)$$

$$\int_0^x I_\varrho(a) a^\varrho \cos(2x-a) da = \frac{x^{\varrho+1}}{2\varrho+1} \{ \cos x I_\varrho(x) - \sin x I_{\varrho+1}(x) \}, \quad R(\varrho) > -\frac{1}{2} \quad (8)$$

2. Voor $\nu = \varrho$ gaat II over in:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{n! \Gamma(\varrho+n+\frac{1}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \frac{\Gamma(-\frac{1}{2})}{\Gamma(\varrho+1)} \left(\frac{x}{2}\right)^{\varrho+\frac{1}{2}} I_{\varrho-\frac{1}{2}}(x) \quad (II')$$

Vervangen we nu in (2) ϱ door $\varrho+n$ en vermenigvuldigen we daarna beide leden met $\frac{\Gamma(n-\frac{1}{2})}{2^{\varrho+n} n! \Gamma(\varrho+n+\frac{1}{2})}$, dan volgt na sommatie over n van 0 tot ∞ onder toepassing van II' in het linkerlid onder het integraalteeken:

$$\left. \begin{aligned} & \int_0^x I_{\nu-\frac{1}{2}}(x-a) I_{\varrho-\frac{1}{2}}(a) (x-a)^{\nu+\frac{1}{2}} a^{\varrho+\frac{1}{2}} da = \\ & = \frac{2^{\nu+\varrho} \Gamma(\nu+1) \Gamma(\varrho+1) x}{\Gamma(-\frac{1}{2}) \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{n! \Gamma(\nu+\varrho+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{\nu+\varrho+n} I_{\nu+\varrho+n}(x), \end{aligned} \right\} \quad (9)$$

$$R(\nu) > -\frac{1}{2}, \quad R(\varrho) > -\frac{1}{2},$$

waarvan het rechterlid in eindigen vorm geschreven kan worden.

Stellen we immers in I $\mu = \nu + \frac{1}{2}$ en vervangen we daarna ν door ϱ , dan volgt:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\varrho + 1)} \left(\frac{x}{2}\right)^{\varrho-\frac{1}{2}} I_{\varrho+\frac{1}{2}}(x) \quad (\text{III})$$

Wegens

$$\Gamma(n + \frac{1}{2}) = (n - \frac{1}{2}) \Gamma(n - \frac{1}{2}) = n \Gamma(n - \frac{1}{2}) - \frac{1}{2} \Gamma(n - \frac{1}{2})$$

is voor het linkerlid van III te schrijven:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{(n-1)! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \\ & = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{5}{2})} \left(\frac{x}{2}\right)^{\varrho+n+1} I_{\varrho+n+1}(x) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x), \end{aligned}$$

dus op grond van III, na vervanging van ϱ door $\varrho + 1$:

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(\varrho + 2)} \left(\frac{x}{2}\right)^{\varrho+\frac{1}{2}} I_{\varrho+\frac{3}{2}}(x) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x),$$

hetgeen ingevuld in III voert tot:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = \frac{2 \Gamma(\frac{1}{2})}{\Gamma(\varrho + 2)} \left(\frac{x}{2}\right)^{\varrho+\frac{1}{2}} \left\{ I_{\varrho+\frac{3}{2}}(x) - \frac{2(\varrho+1)}{x} I_{\varrho+\frac{1}{2}}(x) \right\},$$

of, zoo we op grond van een eigenschap der BESSEL'sche functies substitueeren

$$I_{\varrho+\frac{3}{2}}(x) = \frac{2(\varrho + \frac{1}{2})}{x} I_{\varrho+\frac{1}{2}}(x) - I_{\varrho-\frac{1}{2}}(x):$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(\varrho + n + \frac{3}{2})} \left(\frac{x}{2}\right)^{\varrho+n} I_{\varrho+n}(x) = - \frac{\Gamma(\frac{1}{2})}{\Gamma(\varrho + 2)} \left(\frac{x}{2}\right)^{\varrho-\frac{1}{2}} \{ x I_{\varrho-\frac{1}{2}}(x) + I_{\varrho+\frac{1}{2}}(x) \}, \quad (\text{IV})$$

zoodat (9) in verband met IV, na hierin vooraf ϱ door $\nu + \varrho$ te hebben vervangen, overgaat in:

$$\left. \begin{aligned} & \int_0^x I_{\nu-\frac{1}{2}}(x-a) I_{\varrho-\frac{1}{2}}(a) (x-a)^{\nu+\frac{1}{2}} a^{\varrho+\frac{1}{2}} da = \\ & \frac{\Gamma(\nu+1) \Gamma(\varrho+1)}{\Gamma(\nu+\varrho+2) \sqrt{2\pi}} x^{\nu+\varrho+\frac{1}{2}}(x) \{ x I_{\nu+\varrho-\frac{1}{2}} + I_{\nu+\varrho+\frac{1}{2}}(x) \}, \end{aligned} \right\} \quad (9')$$

$$R(\nu) > -\frac{1}{4}, \quad R(\varrho) > -\frac{1}{2}.$$

Langs anderen weg kwamen we ook reeds tot deze formule¹⁾.

¹⁾ Proc. Kon. Akad. v. Wetensch., Amsterdam, 34, No. 1, 150, (4) (1931).

Op grond van de betrekkingen (a) voert substitutie van $\nu = 0$ in (9') en van $\varrho = \frac{1}{2}$ in (2), zoo we daarna ν door ϱ vervangen, tot:

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \cos(x-a) da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} \{x I_{\varrho-\frac{1}{2}}(x) + I_{\varrho+\frac{1}{2}}(x)\}, R(\varrho) > -\frac{1}{2} \quad (10)$$

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \sin(x-a) da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} I_{\varrho+\frac{1}{2}}(x), R(\varrho) > -\frac{1}{2}. \quad (11)$$

Vermenigvuldigen we nu weer beide leden van (10) met $\sin x$ resp. $\cos x$ en die van (11) met $\cos x$ resp. $\sin x$, dan volgen na aftrekking en optelling der overeenkomstige leden:

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \sin a da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} \{x \sin x I_{\varrho-\frac{1}{2}}(x) + (\sin x - x \cos x) I_{\varrho+\frac{1}{2}}(x)\}, R(\varrho) > -\frac{1}{2} \quad (12)$$

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \cos a da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} \{x \cos x I_{\varrho-\frac{1}{2}}(x) + (\cos x + x \sin x) I_{\varrho+\frac{1}{2}}(x)\}, R(\varrho) > -\frac{1}{2} \quad (13)$$

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \sin(2x-a) da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} \{x \sin x I_{\varrho-\frac{1}{2}}(x) + (\sin x + x \cos x) I_{\varrho+\frac{1}{2}}(x)\}, R(\varrho) > -\frac{1}{2} \quad (14)$$

$$\int_0^x I_{\varrho-\frac{1}{2}}(a) a^{\varrho+\frac{1}{2}} \cos(2x-a) da = \frac{x^{\varrho+\frac{1}{2}}}{2(\varrho+1)} \{x \cos x I_{\varrho-\frac{1}{2}}(x) + (\cos x - x \sin x) I_{\varrho+\frac{1}{2}}(x)\}, R(\varrho) > -\frac{1}{2} \quad (15)$$

Geophysics. — *On Surface Waves in a Stratified Medium.* I. By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS.)

(Communicated at the meeting of March 28, 1942.)

Par. 1. Introduction.

The first theory evolved by the seismologists to explain the general features of the seismic data, in particular the shape of the seismograms, presupposed a homogeneous medium, bounded by a plane surface. In such a medium it is possible for three kinds of waves to exist, namely the longitudinal, the transversal, and the surface (RAYLEIGH-) waves; the velocity of these waves is independent of the frequency of the vibrations. This theory was completely worked out by LAMB¹⁾ in 1904, who derived from it the first theoretically calculated seismogram. This seismogram, however, differs in two important ways from the registered seismograms; firstly: the horizontal surface displacement observed in earthquakes is often much larger than the vertical displacement, which is in contradiction to the result arrived at by LAMB; secondly: several features of the seismograms appear to point to a dispersive character of the wave propagation, but no dispersion can occur in a single elastic medium bounded by a plane surface. LAMB concluded, therefore, that it is impossible for "almost any conceivable theory", based on the above assumption, to explain the seismic data.

The second important step in this theoretical investigation was taken by LOVE in 1911; LOVE²⁾ found that in a homogeneous medium covered by a plane layer of other material, a second type of surface waves can exist. As these waves give rise only to a horizontal displacement and as the wave velocity is dependent on the frequency of the vibrations, it is possible for this theory to meet the two difficulties to the first theory. The velocity equation of these LOVE waves being very simple, the only remaining problem is to assume the depth of the layer in such a way that the theory is in accordance with the observations. In recent years it appeared to be necessary to suppose the existence of a second surface layer. This problem, i.e. the determination of the superficial structure of the earth from the seismic observations, is still one of the most important parts of the analysis of seismograms.

In his book "Some Problems of Geodynamics" LOVE has also shown that in a stratified medium other surface waves, not of the LOVE type, can occur. Owing to the complicated velocity equation for this type of waves, the enquiry into the existence and the properties of these waves has not led very far.

Already in 1898 BROMWICH³⁾ had studied a very simplified form of this equation in the special case of an incompressible semi-infinite body, covered by a very thin layer of other (also incompressible) material. In this case the equation appears to be identical with the RAYLEIGH equation, with a small correction term, the waves being of the RAYLEIGH type with a slightly altered velocity.

The enquiry made by LOVE extended to the case of a surface layer the depth of which is large in comparison with the wave-length, the two media being incompressible. The equation can then be decomposed into two other equations; the former of which is the velocity equation of RAYLEIGH waves transmitted through a semi-infinite body composed by the same material as that in the surface layer. With respect to the second equation LOVE arrives at the erroneous conclusion that this equation is not important, as a root is only possible for a small range of values of the material constants which do not occur in actual conditions.

Supposing the layer of infinite thickness STONELEY⁴⁾ deduced in 1924 an equation relating to waves which are mainly propagated along the surface of separation between the two media. This equation is identical with the second of the equations just mentioned,

but as STONELEY arrived at the same erroneous conclusion with respect to these waves as LOVE, the investigation was then not further pursued.

In a previous paper⁵⁾ the author showed that these STONELEY waves are possible for widely different values of the material constants (see also SEZAWA⁶⁾).

In 1934 it was shown by STONELEY that also in the case of two compressible media the general velocity equation can be split up into two other equations if the wave-length is very small in comparison with the depth of the layer⁷⁾. These equations are of course the STONELEY equation and the RAYLEIGH equation for the layer.

Hence we can expect that there are three kinds of surface waves possible in a stratified medium:

1. the LOVE waves;
2. the generalised STONELEY waves, which must exist if the layer has a very large thickness and which are then nearly identical with the STONELEY waves;
3. the generalised RAYLEIGH waves, which exist if the layer is very thick (\approx RAYLEIGH waves in the layer, as found by LOVE) and also if the layer is very thin (\approx RAYLEIGH waves in the subjacent medium, as shown by BROMWICH).

A more general investigation of the equation of the generalised RAYLEIGH and STONELEY waves was meanwhile attempted by SEZAWA⁸⁾, who, however, used only numerical methods. In the publications of SEZAWA and KANAI^{9, 10)} in 1938 and 1939 the wave velocity of the possible surface waves of the second and third types was calculated for widely different values of the coefficients of rigidity of the two media and for every value of the depth of the layer. It is evident that this method is only to be applied if a general treatment is impossible and that it is very difficult in this way to get a general survey of the possible roots of the equation.

In the present paper the whole problem of the possible types of surface waves in a stratified medium is dealt with again. If we suppose an incident wave being propagated in the underlying medium, the following wave systems are possible:

1. if the incident wave is transversal, vibrating perpendicular to the plane of incidence, there will occur a reflected wave in this medium, a refracted wave in the layer, and a reflected wave in the layer. All these 4 waves are of course transversal.
2. If the incident wave is longitudinal or transversal, vibrating in the plane of incidence, there exists a wave system composed of a longitudinal and a transversal reflected wave in the subjacent medium, as well as longitudinal and transversal refracted and reflected waves in the layer; therefore in total 7 waves.

As there are in the first case 3 boundary conditions, in the second case 6, the amplitudes of all waves can be expressed in the amplitude of the incident wave.

Now it is possible that there are only 3 — respectively 6 — waves in such a system, one of the amplitudes being equal to zero. The determinant of the coefficients of the amplitudes figuring in the boundary conditions then must be equal to zero; the angle of incidence at which such a particular wave-system occurs is determined by this equation.

If we put the amplitude of the incident wave equal to zero, we obtain in both cases a peculiar wave-system. In the first case this system will appear to be the LOVE wave-system, the determinant equation being the LOVE equation. In the second case the wave system is more complicated as is also the determinant equation; as will be shown the waves we obtain here are surface waves and the corresponding equation is the same as the equation of the generalised RAYLEIGH and STONELEY waves.

In the second paragraph of the present paper this derivation of the LOVE wave-system will be given with a very short discussion of the equation. As this equation is well known this paragraph is only inserted for the sake of completeness.

In Par. 3 the more complicated case will be treated; the determinant equation will be expanded and a preliminary reduction of this equation will be effectuated. At the same time it will be shown that this equation is identical with the equation of the generalised *R* and *S* waves.

It will be obvious that this general equation in special cases must result in the more simple RAYLEIGH and STONELEY equation. A new special case is obtained if we put the density of the underlying medium equal to zero, in which case the equation has to be reduced to the wave equation of an isolated layer. The R and S equation having already been discussed an investigation must be made into the properties of the waves in an isolated layer, which is carried out in Par. 4.

In the next paragraphs the analyses of the generalised R and S wave equation will be given; in Par. 5 those values of the material constants will be determined for which a generalised R or S wave system can exist and in Par. 6 the general shape of the dispersion curves, determining the wave-velocity as a function of the wave-length, will be derived.

Par. 2. The LOVE waves.

Supposing (see figure 1) the incident wave \mathfrak{U}_e transversal vibrating perpendicular to

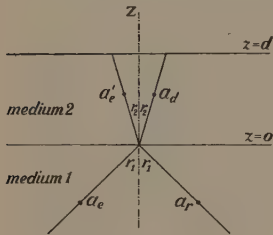


Fig. 1

the plane of incidence with the frequency $\frac{P}{2\pi}$, we have the following waves:

$$\mathfrak{U}_e \cdot e^{i(pt - k_1 x \sin r_1 - k_1 z \cos r_1)}, \quad \mathfrak{U}_r \cdot e^{i(pt - k_1 x \sin r_1 + k_1 z \cos r_1)} \\ \mathfrak{U}'_e \cdot e^{i(pt - k_2 x \sin r_2 + k_2 z \cos r_2)}, \quad \mathfrak{U}_d \cdot e^{i(pt - k_2 x \sin r_2 - k_2 z \cos r_2)}$$

where $k = \frac{P}{\mathfrak{B}}$, being the velocity of transversal waves.

The boundary conditions are:

continuity of displacement at $z = 0$:

$$\mathfrak{U}_e + \mathfrak{U}_r = \mathfrak{U}'_e + \mathfrak{U}_d$$

continuity of tension at $z = 0$:

$$-\mathfrak{U}_e \cdot \sin 2r_1 + \mathfrak{U}_r \cdot \sin 2r_1 = \varrho_2/\varrho_1 \cdot \mathfrak{U}'_e \sin 2r_2 - \varrho_2/\varrho_1 \cdot \mathfrak{U}_d \sin 2r_2$$

where $\varrho =$ density

tension = 0 at the free surface:

$$\mathfrak{U}'_e e^{ik_2 d \cos r_2} = \mathfrak{U}_d \cdot e^{-ik_2 d \cos r_2};$$

or

$$\left\{ \begin{array}{l} \mathfrak{U}_e + \mathfrak{U}_r - \mathfrak{U}'_e (1 + e^{2i\alpha}) = 0 \\ + \mathfrak{U}_e - \mathfrak{U}_r + \frac{\varrho_2 \sin 2r_2}{\varrho_1 \sin 2r_1} \cdot \mathfrak{U}'_e (1 - e^{2i\alpha}) = 0, \text{ with } \alpha = k_2 d \cos r_2. \end{array} \right.$$

A case of special reflection occurs if we put $\mathfrak{U}_r = 0$; then

$$\left| \begin{array}{cc} +1 & -(1 + e^{2i\alpha}) \\ +1 & + \frac{\varrho_2 \sin 2r_2}{\varrho_1 \sin 2r_1} (1 - e^{2i\alpha}) \end{array} \right| = 0$$

or

$$\frac{\varrho_1 \sin 2r_1}{\varrho_2 \sin 2r_2} = i \tan \alpha.$$

Another special wave-system is obtained if we put \mathfrak{U}_e equal to zero; the determinant equation then becomes:

$$\frac{\varrho_1 \sin 2r_1}{\varrho_2 \sin 2r_2} = -i \tan \alpha.$$

Both equations can only be solved if $\cos r_1$ is imaginary; as the waves have to decrease with increasing distance to the surface of separation, $\cos r_1$ must be taken positive imaginary in the first case ($\mathfrak{A}_r = 0$) and negative imaginary in the case of $\mathfrak{A}_e = 0$.

Putting $\sin r_1 = \frac{1}{\sqrt{\zeta}}$ ($\zeta < 1$) we get in both cases: $\frac{\mu_2}{\mu_1} \sqrt{\frac{1-\zeta}{\omega\zeta-1}} = \tan a$, which is identical with the equation of LOVE (μ being the coefficient of rigidity and $\omega = \frac{\mathfrak{B}_1^2}{\mathfrak{B}_2^2}$).

For further discussion of this wave-system and period-equation we may refer to LOVE's "Some problems of geodynamics".

Par 3. The generalised RAYLEIGH and STONELEY waves.

If we suppose (see figure 2) the incident wave longitudinal and being propagated in the underlying medium 1 the following waves can exist:

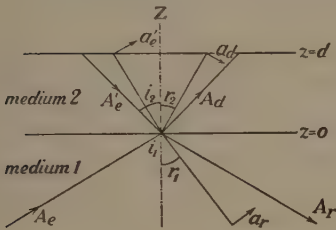


Fig. 2

in the superficial layer

$$A_e' \cdot e^{i(pt-h_2x \sin i_2+h_2z \cos i_2)}, \quad A_d' \cdot e^{i(pt-h_2x \sin i_2-h_2z \cos i_2)}$$

$$\mathfrak{A}_e' \cdot e^{i(pt-k_2x \sin r_2+k_2z \cos r_2)}, \quad \mathfrak{A}_d' \cdot e^{i(pt-k_2x \sin r_2-k_2z \cos r_2)}$$

and in medium 1

$$A_e \cdot e^{i(pt-h_1x \sin i_1-h_1z \cos i_1)}, \quad A_r \cdot e^{i(pt-h_1x \sin i_1+h_1z \cos i_1)}$$

$$\mathfrak{A}_r \cdot e^{i(pt-k_1x \sin r_1+k_1z \cos r_1)}.$$

The boundary conditions are:
the tension at the free surface $z = 0$ is equal to zero

$$\begin{cases} n_2 A_d e^{-i\alpha'} \cos 2r_2 - \mathfrak{A}_d e^{-i\beta'} \sin 2r_2 + n_2 A_e' e^{+i\alpha'} \cos 2r_2 - \mathfrak{A}_e' e^{+i\beta'} \sin 2r_2 = 0 \\ A_d e^{-i\alpha'} \sin 2i_2 + n_2 \mathfrak{A}_d e^{-i\beta'} \cos 2r_2 - A_e' e^{+i\alpha'} \sin 2i_2 - n_2 \mathfrak{A}_e' e^{+i\beta'} \cos 2r_2 = 0 \end{cases}$$

continuity of movement at $z = 0$

$$\begin{cases} A_e \sin i_1 + A_r \sin i_1 + \mathfrak{A}_r \cos r_1 = A_d \sin i_2 + \mathfrak{A}_d \cos r_2 + A_e' \sin i_2 + \mathfrak{A}_e' \cos r_2 \\ A_e \cos i_1 - A_r \cos i_1 + \mathfrak{A}_r \sin r_1 = A_d \cos i_2 - \mathfrak{A}_d \sin r_2 - A_e' \cos i_2 + \mathfrak{A}_e' \sin r_2 \end{cases}$$

continuity of tension at $z = 0$

$$\left\{ \begin{aligned} A_e \cos 2r_1 + A_r \cos 2r_1 - \mathfrak{A}_r \frac{\sin 2r_1}{n_1} &= \frac{\rho_2 V_2}{\rho_1 V_1} A_d \cos 2r_2 - \\ &\quad - \frac{\rho_2 \mathfrak{B}_2}{\rho_1 V_1} \mathfrak{A}_d \sin 2r_2 + \frac{\rho_2 V_2}{\rho_1 V_1} A_e' \cos 2r_2 - \frac{\rho_2 \mathfrak{B}_2}{\rho_1 V_1} \mathfrak{A}_e' \sin 2r_2 \\ A_e \sin 2i_1 - A_r \sin 2i_1 - \mathfrak{A}_r \cdot n \cos 2r_1 &= \frac{\mu_2 V_1}{\mu_1 V_2} A_d \sin 2i_2 + \\ &\quad + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \mathfrak{A}_d \cos 2r_2 - \frac{\mu_2 V_1}{\mu_1 V_2} A_e' \sin 2i_2 - \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \mathfrak{A}_e' \cos 2r_2 \end{aligned} \right.$$

where $h = \frac{p}{V}$, V being the velocity of longitudinal waves

$$\text{and } \alpha' = h_2 d \cos i_2, \quad \beta' = k_2 d \cos r_2, \quad n = \frac{V}{\mathfrak{B}}.$$

The special wave-system in which we are interested, is obtained by putting again $A_e = 0$. This system is only possible if:

$$\begin{vmatrix} 0 & 0 & n_2 e^{-i\alpha'} \cos 2r_2 & -e^{-i\beta'} \sin 2r_2 & n_2 e^{+i\alpha'} \cos 2r_2 & -e^{+i\beta'} \sin 2r_2 \\ 0 & 0 & e^{-i\alpha'} \sin 2i_2 & n_2 e^{-i\beta'} \cos 2r_2 & -e^{+i\alpha'} \sin 2i_2 & -n_2 e^{+i\beta'} \cos 2r_2 \\ \sin i_1 & \cos r_1 & -\sin i_2 & -\cos r_2 & -\sin i_2 & -\cos r_2 \\ -\cos i_1 & \sin r_1 & -\cos i_2 & +\sin r_2 & +\cos i_2 & -\sin r_2 \\ \cos 2r_1 & -\frac{\sin 2r_1}{n_1} & -\frac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2r_2 + \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2r_2 & -\frac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2r_2 + \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2r_2 & -\frac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2r_2 + \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2r_2 & -\frac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2r_2 + \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2r_2 \\ -\sin 2i_1 & -n_1 \cos 2r_1 & -\frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 - \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 + \frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 & -\frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 - \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 + \frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 & -\frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 - \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 + \frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 & -\frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 - \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 + \frac{\mu_2 V_1}{\mu_1 V_2} \sin 2i_2 + \frac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2r_2 \end{vmatrix} = 0$$

which can be reduced to

$$\begin{aligned} & -(P + Q_2) \{ n_2^2 \cos^2 2r_2 (e^{-i\alpha'} - e^{+i\alpha'}) (e^{-i\beta'} - e^{+i\beta'}) + \sin 2i_2 \sin 2r_2 (e^{-i\alpha'} + e^{+i\alpha'}) (e^{-i\beta'} + e^{+i\beta'}) \} \\ & -(S + Q_1) \{ n_2^2 \cos^2 2r_2 (e^{-i\alpha'} + e^{+i\alpha'}) (e^{-i\beta'} + e^{+i\beta'}) + \sin 2i_2 \sin 2r_2 (e^{-i\alpha'} - e^{+i\alpha'}) (e^{-i\beta'} - e^{+i\beta'}) \} \\ & + R_1 \{ n_2^2 \cos^2 2r_2 (e^{-i\alpha'} + e^{+i\alpha'}) (e^{-i\beta'} - e^{+i\beta'}) + \sin 2i_2 \sin 2r_2 (e^{-i\alpha'} - e^{+i\alpha'}) (e^{-i\beta'} + e^{+i\beta'}) \} \\ & + R_2 \{ n_2^2 \cos^2 2r_2 (e^{-i\alpha'} - e^{+i\alpha'}) (e^{-i\beta'} + e^{+i\beta'}) + \sin 2i_2 \sin 2r_2 (e^{-i\alpha'} + e^{+i\alpha'}) (e^{-i\beta'} - e^{+i\beta'}) \} \\ & + 8 n_2 \cos 2r_2 \sqrt{\sin 2i_2 \sin 2r_2} \cdot (\sqrt{P Q_1} - \sqrt{S Q_2}) = 0. \end{aligned}$$

We have used here the same abbreviations as in our previous paper⁵⁾, namely:

$$\begin{aligned} P &= n_1 n_2 \sin^2 r_2 \left(\frac{\varrho_2}{\varrho_1} \cos 2r_2 - \cos 2r_1 \right)^2, \quad S = 4 \cos i_1 \cos i_2 \cos r_1 \cos r_2 \cdot \sin^2 r_1 \left(1 - \frac{\mu_2}{\mu_1} \right)^2 \\ Q_1 &= n_1 \cos i_2 \cos r_2 \left(\cos 2r_1 + \frac{\mu_2}{\mu_1} \sin^2 r_1 \right)^2, \quad Q_2 = n_2 \cos i_1 \cos r_1 \left(\cos 2r_2 + 2 \frac{\mu_1}{\mu_2} \sin^2 r_2 \right)^2 \cdot \frac{\varrho_2}{\varrho_1} \frac{\mu_2}{\mu_1} \\ R_1 &= \frac{\mu_2}{\mu_1} \cdot \frac{\sin i_1 \cos i_2 \cos r_1}{\sin r_2}, \quad R_2 = \frac{\varrho_2}{\varrho_1} \cdot \frac{\sin i_2 \cos i_1 \cos r_2}{\sin r_1}. \end{aligned}$$

Using the identities:

$$\begin{aligned} \sqrt{P \sin 2i_2 \sin 2r_2} - n_2 \cos 2r_2 \sqrt{Q_1} &= -n_2 \cos 2r_1 \sqrt{n_1 \cos i_2 \cos r_2} \\ \sqrt{Q_2 \sin 2i_2 \sin 2r_2} + n_2 \cos 2r_2 \sqrt{S} &= 2 n_2 \sin r_1 \sqrt{\cos i_1 \cos r_1 \cos i_2 \cos r_2} \end{aligned}$$

the equation becomes

$$\begin{aligned} & (P + Q_2) \{ n_2^2 \cos^2 2r_2 \operatorname{tg} \alpha' \operatorname{tg} \beta' - \sin 2i_2 \cdot \sin 2r_2 \cdot \varphi \} + \\ & + (S + Q_1) \{ -n_2^2 \cos^2 2r_2 \cdot \varphi + \sin 2i_2 \cdot \sin 2r_2 \operatorname{tg} \alpha' \operatorname{tg} \beta' \} = \\ & i R_1 \{ n_2^2 \cos^2 2r_2 \operatorname{tg} \beta' + \sin 2i_2 \cdot \sin 2r_2 \cdot \operatorname{tg} \alpha' \} + \\ & + i R_2 \{ n_2^2 \cos^2 2r_2 \cdot \operatorname{tg} \alpha' + \sin 2i_2 \cdot \sin 2r_2 \cdot \operatorname{tg} \beta' \} \\ & + \frac{n_2^2}{n_1} \cos i_2 \cos r_2 \{ n_1^2 \cos^2 2r_1 + \sin 2i_1 \sin 2r_1 \}, \end{aligned}$$

where

$$\varphi = 1 - \frac{1}{\cos \alpha' \cos \beta'}.$$

As can be easily demonstrated, this equation can only be solved if $\cos i_1$ and $\cos r_1$ are imaginary; the amplitudes of the waves in the subjacent medium decrease therefore exponentially with increasing distance to the surface of separation (taking the two cosines of course negative imaginary). Consequently these waves are surface waves in the same sense as the LOVE waves. Further it is possible that $\cos i_2$ is also imaginary or that $\cos i_2$ and $\cos r_2$ are both imaginary. In the last case all waves in the two media are damped.

As we may expect that the equation has more than one solution if one of its terms is a cyclic function, it is advisable to start with the case of all cosines being imaginary, which give rise to hyperbolic functions only. Moreover these completely damped wave-systems must be closely connected with the important RAYLEIGH and STONELEY waves, as these systems are also completely damped.

In accordance with the method adopted in our paper above mentioned we put $\sin r_1 = \frac{1}{\sqrt{-\zeta}}$; then

$$\sin i_1 = \frac{n_1}{\sqrt{-\zeta}}, \quad \sin i_2 = \frac{m_1}{\sqrt{-\zeta}}, \quad \sin r_2 = \frac{m_1}{\sqrt{-\zeta}}$$

with

$$n_1 = \frac{V_1}{\mathfrak{B}_1}, \quad m_1 = \frac{V_2}{\mathfrak{B}_1} \quad \text{and} \quad m_2 = \frac{\mathfrak{B}_2}{\mathfrak{B}_1};$$

hence

$$\cos r_1 = -i \sqrt{\frac{1-\zeta}{\zeta}}, \quad \cos i_1 = -in_1 \sqrt{\frac{1-r_1\zeta}{\zeta}}, \quad \cos r_2 = im_2 \sqrt{\frac{1-\omega\zeta}{\zeta}}, \quad \cos i_2 = im_1 \sqrt{\frac{1-\gamma\zeta}{\zeta}},$$

where

$$v_1 = \frac{\mathfrak{B}_1^2}{V_1^2} = \frac{1}{\lambda_1/\mu_1 + 2}; \quad \omega = \frac{\mathfrak{B}_1^2}{\mathfrak{B}_2^2} = \frac{\mu_1 \varrho_2}{\mu_2 \varrho_1}; \quad \gamma = \frac{\mathfrak{B}_1^2}{V_2^2} = \omega v_2 = \omega \cdot \frac{1}{\lambda_2/\mu_2 + 2}.$$

The equation becomes:

$$\left. \begin{aligned} & (P - Q_2) \cdot \{ 4 \sqrt{(1-\omega\zeta)(1-\gamma\zeta)} \cdot \varphi - (2-\omega\zeta)^2 \cdot \operatorname{tgh} a \operatorname{tgh} \beta \} + \\ & + (S - Q_1) \cdot \{ 4 \sqrt{(1-\omega\zeta)(1-\gamma\zeta)} \cdot \operatorname{tgh} a \operatorname{tgh} \beta - (2-\omega\zeta)^2 \cdot \varphi \} = \\ & = R_1 \cdot \{ 4 \sqrt{(1-\omega\zeta)(1-\gamma\zeta)} \cdot \operatorname{tgh} a - (2-\omega\zeta)^2 \cdot \operatorname{tgh} \beta \} + \\ & + R_2 \cdot \{ 4 \sqrt{(1-\omega\zeta)(1-\gamma\zeta)} \cdot \operatorname{tgh} \beta - (2-\omega\zeta)^2 \cdot \operatorname{tgh} a \} + \\ & + \omega^2 \zeta^2 \frac{\sqrt{(1-\omega\zeta)(1-\gamma\zeta)}}{\cosh a \cosh \beta} \{ 4 \sqrt{(1-\zeta)(1-r_1\zeta)} - (2-\zeta)^2 \}, \end{aligned} \right\} \quad (1)$$

where

$$\alpha = \frac{pd}{\mathfrak{B}_1} \sqrt{\frac{1-\gamma\zeta}{\zeta}}, \quad \beta = \frac{pd}{\mathfrak{B}_1} \sqrt{\frac{1-\omega\zeta}{\zeta}}, \quad \varphi = 1 - \frac{1}{\cosh a \cosh \beta},$$

and P, S, Q_1, Q_2, R_1, R_2 are the terms of the STONELEY equation $P - Q_2 + S - Q_1 = R_1 + R_2$, namely:

$$\begin{aligned} P &= (1 - 2\mu_2/\mu_1 + \varrho_2/\varrho_1 \zeta)^2, \quad S = (2 - 2\mu_2/\mu_1)^2 \cdot \sqrt{(1-\zeta)(1-r_1\zeta)(1-\omega\zeta)(1-\gamma\zeta)}, \\ Q_1 &= (2 - 2\mu_2/\mu_1 - \zeta)^2 \cdot \sqrt{(1-\omega\zeta)(1-\gamma\zeta)}, \quad Q_2 = (2 - 2\mu_2/\mu_1 + \varrho_2/\varrho_1 \zeta)^2 \cdot \sqrt{(1-\zeta)(1-r_1\zeta)}, \\ R_1 &= \varrho_2/\varrho_1 \zeta^2 \sqrt{(1-\zeta)(1-\gamma\zeta)}, \quad R_2 = \varrho_2/\varrho_1 \zeta^2 \sqrt{(1-\omega\zeta)(1-r_1\zeta)}. \end{aligned}$$

It is to be remarked that this equation is identical with the equation derived by SEZAWA, if we put the function f , occurring in his notation, equal to our function k_1^2/ζ .

In the following special cases the reduction of equation (1) will be obvious:

1. If

$$N = \frac{pd}{\mathfrak{B}_2} = 0,$$

in other words if the depth of the layer is infinitely small in comparison with the transverse wave-length $\left(= \frac{2\pi\mathfrak{B}_2}{p} \right)$, the equation can be reduced to

$$4\sqrt{(1-\zeta)(1-r_1\zeta)} - (2-\zeta)^2 = 0,$$

the RAYLEIGH equation of medium 1. ($\varphi = \tanh \alpha = \tanh \beta = 0$.)

2. If $N = \infty$ then $\varphi = \tanh \alpha = \tanh \beta = 1$; equation (1) becomes

$$\{4\sqrt{(1-\omega\zeta)(1-\gamma\zeta)} - (2-\omega\zeta)^2\} \{P - Q_2 + S - Q_1 - R_1 - R_2\} = 0:$$

the RAYLEIGH equation of medium 2 and the STONELEY equation.

$$3. \text{ If } \frac{\varrho_2}{\varrho_1} = \frac{\mu_2}{\mu_1} = 0, \quad \omega = \frac{\varrho_2\mu_1}{\varrho_1\mu_2} \text{ remaining finite}$$

we get

$$(2-\eta)^4 \tanh \alpha \tanh \beta - 8(2-\eta)^2 \sqrt{(1-\eta)(1-r_2\eta)} \cdot \varphi + 16(1-\eta)(1-r_2\eta) \tanh \alpha \tanh \beta = 0$$

where

$$\eta = \omega\zeta, \alpha = N\sqrt{\frac{1-r_2\eta}{\eta}} \text{ and } \beta = N\sqrt{\frac{1-\eta}{\eta}}.$$

It is evident that in this case ($\mu_1 = \varrho_1 = 0$) the subjacent medium does not exist; this equation must therefore relate to the vibrations of an isolated layer. We shall investigate in the next paragraph the problem of the wave-systems occurring in an elastic plate, as it has an important bearing on the general equation (1).

(To be continued.)

Biochemistry. — *Coexisting complex coacervates.* By H. G. BUNGENBERG DE JONG and E. G. HOSKAM. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of February 28, 1942.)

1. *Introduction.*

We have previously described how two coexisting complex coacervates are formed in mixtures of gelatine, gum arabic and Na-Nucleinate sols in certain mixing proportions with sufficient pH reduction¹⁾. The results were set out in a ternary diagram which showed that the area of mixing proportions in which there are two coacervates is roughly between the mixing proportions located on the sides of the triangle in which the reversal of charge lies in the two systems gelatine + gum arabic and gelatine + Na-Nucleinate.

The problem of the significance of the charge for the formation of coexisting coacervates is again discussed in the following pages. We were especially interested in the course of the lines connecting the coexisting coacervates in the area of the two coexisting coacervates.

2. *Material and technique.*

In the previous investigation we made use of unpurified colloid preparations, but for this investigation we used them purified, viz. isoelectric gelatine, Na-Arabinate and Na-Yeast nucleinate, the preparation of which has been described elsewhere²⁾. In the following pages we refer to these preparations as G (gelatine), A (Na-Arabinatie), and N (Na-Nucleinate).

Of these 3 preparations we prepared stock sols by dissolving 5 g. air-dry samples in 250 g. dist. water. These stock sols were preserved in the refrigerator for future use.

In the previous investigation the pH reduction was caused by diluted acetic acid, in the present investigation acetate buffers were used. As neutral salts, however neutralize the complex coacervation it is recommendable to keep the Na-Acetate final concentration in this buffer rather low, for which we chose 10 m aeq. p. L. To 10 cc stock sol or mixture of stock sols we always added 5 cc buffer, the composition of which was as follows: 30 cc Na-Acetate 0.1 N + 50 cc acetic acid 1 N, dist. water being added until 100 cc. For the three stock sols separately (H electrode at 40°) the pH was then: G = 3.65, A = 3.57; and Na = 3.76. So the three buffered sols are not exactly, but approximately isohydric, which is not to be wondered at, as only a comparatively slight Na-Acetate concentration was admissible, so that better buffering was not to be expected with the comparatively great colloid concentrations. In the area of mixing proportions, in which 2 coexisting coacervates are formed (extending between ca. 50% A + 50% G with pH = 3.61 and between ca. 30% N + 70% G with pH = 3.68) the pH does not vary quite 0.1 pH.

3. *The coacervation areas.*

First we investigated the coacervation in the binary combinations gelatine + Na-

1) H. G. BUNGENBERG DE JONG and A. DE HAAN, *Biochem. Z.* **263**, 33 (1933).

2) Isoelectric gelatine, prepared from gelatine F00 extra of the "Lijm- en Gelatine-fabriek 'Delft'" at Delft. Method of preparation see *Koll. Beihefte*, **43**, 256 (1936).

Na-Arabinate prepared from gomme Senegal pepite boule blanche I of ALLAND et ROBERT, Paris (preparation see *Kolloid Beihefte* **47**, 254 (1938)).

Na-Nucleinate prepared from N.-Nucleinate of E. MERCK (preparation see *Kolloid Beihefte* **47**, 254 (1938)).

Arabinate, resp. gelatine + Na-Nucleinate at 40° C. Therefore the following series of mixtures were prepared in sedimentation tubes:

a cc A + (10—a) cc G + 5 cc buffer (I)

resp.

a cc N + (10—a) cc G + 5 cc buffer (II)

In which G, A and N stand for the stock sols mentioned in 2 (5 G air dry colloid + 250 cc dist. water). The sedimentation tubes were left in the thermostat at 40° till the following morning, when the coacervate volumes were read in 0.1 cc, namely:

Combination G + A		Combination G + N	
a	vol.	a	vol.
2	0.2	1.5	0.2
3	3.4	2	2.4
4	6.8	2.5	3.9
4.5	8.3	3.5	3.9
5	9.1	4.5	3.8
5.5	9.2	5	3.4
6	8.6	6	3.0
7	4.8	8	1.1
8	0.1	9	0.2

Graphically we found that for series I the coacervation takes place between 19% and 81% A, for series II between 9% N and 93% N.

Electrophoretic measurements at 40° gave the following points of charge reversal for these series:

48% A for series I and 27% N for series II.

Subsequently larger quantities of these 48% A resp. 27% N mixtures were made and with these a series of mixtures of the following composition:

a cc (48% A) + (10—a) cc (27% N) + 5 cc buffer (III)

was prepared in sedimentation tubes and the coacervate layers were noted down after ca. 40 hours in the thermostat.

The results are pictured in Fig. 1.

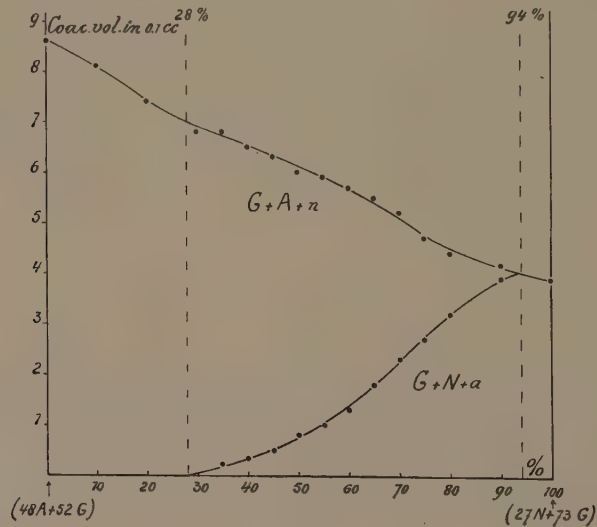


Fig. 1.

From this it is seen that in a certain section of mixing proportions (expressed in % of the N + G system this section extends from 28 % to 94 %), two coexisting coacervates are formed. The one with the greatest specific gravity and the highest nucleinate percentage is indicated in the figure as G + N + a. To the right of the dotted line it passes without any interruption into the G + N coacervate. The complex coacervate of less specific gravity and high arabinates percentage is indicated in the figure as G + A + n, to the left of the dotted line it passes without interruption into the G + A coacervate.

In the following survey the boundaries are indicated of the area in which coexisting coacervates occur, expressed in % of the system indicated as second system. The series III, IV, V, VI and VII were obtained from determinations of the coacervate volume curves (analogous to fig. 1). In the case of VIII and IX we followed a different method, in which a number (climbing up with 1 % of the mixing proportion in the critical area) of mixtures was prepared and microscopically investigated. In order accurately to determine the boundary it is necessary to keep the mixtures belonging to the critical area in the thermostat at 40° for at least one hour.

Mixing series No.	Composition of the two systems combined		Mixing section in which coexisting coacervates occur, expressed in % of the 2nd system ¹⁾
	1st System	2nd System	
III	48 A + 52 G	27 N + 73 G	28(a) — 94(n)
IV	30 A + 70 G	85 N + 15 A	11.5(a) — 26.5(n)
V	75 A + 25 G	25 N + 75 G	22(a) — 96(n)
VI	15 N + 85 A	17 N + 83 G	48(a) — 93(n)
VII	10 N + 90 A	10 N + 90 G	35(n) — 75(a)
VIII	100 G	30 N + 70 A	35(n) — 58(a)
IX	100 G	60 N + 40 A	26(a) — 43(?)

With the assistance of these data we have drawn in Fig. 2 the closed curve within which coexisting coacervates occur.

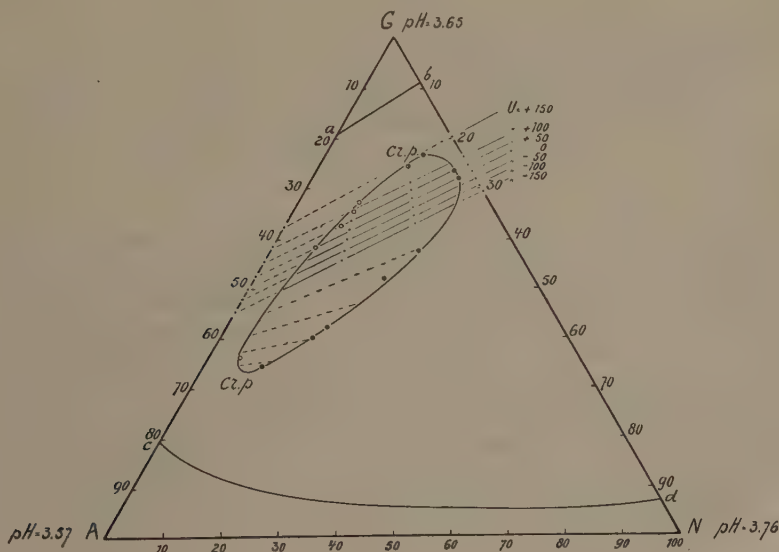


Fig. 2.

¹⁾ The significance of (a) and (n) is explained in § 5.

Moreover curves ab and cd have been drawn which indicate the boundaries of the coacervation. So coacervation does not take place in area Gab, nor in area AcdN.

In area abdc coacervation does take place, one coacervate occurring outside the closed curve (containing the three colloids), two coexisting coacervates within the closed curve, of which the one has a high A and a low N percentage, the other a high N and a low A percentage.

The results described thus far agree very well with the results obtained previously with unpurified colloids and without buffers.

4. Electrophoretic measurements.

We have pointed out in our former publication that the location of the area of the coexisting coacervates extends approximately between the points of charge reversal of the combinations $G + A$ resp., $G + N$ on the sides of the triangle, but accurate measurements were not made at the time.

Now, however correct measurements have been made in a microscopic electrophoresis cuvette at 40° , in which, after a short time of centrifuging of the coacervated system, we suspended a little quartz powder in the equilibrium liquid and measured the electrophoresis velocity of the quartz particles. These measurements were made of four mixing series¹⁾.

The following survey gives the two systems, combined each times and the mixing proportions with which a certain electrophoresis velocity is attained.

Composition of the two systems combined		Mixing proportions in % of the second system, in which the electrophoresis velocity (U) indicated is reached ²⁾						
1st System	2nd System	+150	+100	+50	0	-50	-100	-150
100 G	100 A	38	42	45	48	50.5	52.5	55
100 G	70 A + 30 N		34.9	36.8	38.5	40.4	42.7	—
100 G	40 A + 60 N		29	30.5	32.2	34	35.7	37.3
100 G	100 N	20	24.1	25.7	27.1	28.5	29.8	31.1

The results have been drawn in Fig. 2. The points belonging to the reverse of charge ($U = 0$) show that the uncharged systems within the plane of the triangle lie practically on the straight line connecting the points of charge reversal of the two sides of the triangle $G + A$ and $G + N$. The line divides the plane of the triangle into two parts, in an upper half until vertex G, in which the systems are positive, and in a lower half in which the systems are negative.

This line of reverse of charge intersects the area of the coexisting coacervates, giving a confirmation of what we have said before: The mixability of complex coacervate $G + N$ and $G + A$ is especially slight with the optimal mixing proportions of G and N, resp. of G and A, i.e. there where the compensation of opposed charges is optimal.

From the fact that the line of reverse of charge intersects the area of the coexisting coacervates asymmetrically, into a smaller positive and a larger negative part it would seem that the mutual mixability of the negative $G + A$ and $G + N$ coacervates is smaller than that of the positive coacervates. We cannot as yet account for this fact. In the following section we shall discuss the significance of lines of constant electrophoresis velocity.

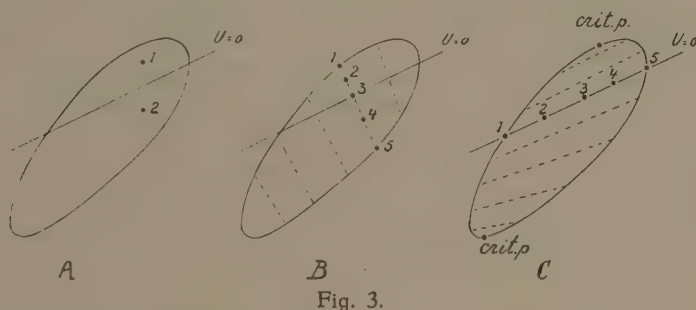
¹⁾ In this place we thank Dr. H. L. BOOY for his assistance in performing the measurements.

²⁾ The electrophoresis velocity is expressed here in arbitrarily chosen units. For details regarding method of the measurements see H. G. BUNGENBERG DE JONG and P. H. THEUNISSEN, *Recueil des Trav. Chim. d. Pays Bas*, **54**, 460 (1935).

5. What is the colloid composition of the coexisting complex coacervates?

This question might of course be answered at once by analyzing the coexisting coacervates. The difficulty arises, however, that although microscopically coexisting coacervates can clearly be distinguished¹⁾ macroscopically the coalescence to separate layers is generally far from easy, with some mixing proportions for instance, the coacervate of high N percentage persists in a division into small drops in the coacervate of high A percentage. Centrifuging is often not sufficient. For the present, therefore, we have to be satisfied with an indirect answer to the question asked.

As described in previous publications the coacervate drops of high A percentage take up those of high N percentage, so that microscopically composite drops are observed. In the electric field these composite drops behave differently according as the mixture



is chosen in the positive part (Fig. 3a, point 1) or in the negative one (Fig. 3a, point 2) of the area of the coexisting coacervates.

When the composite drop is positive (which appears from the cataphoretic direction of the composite drops) the drop of high Na percentage enclosed within the drop of high A percentage also moves into the direction in which the drop of high A percentages electrophoretizes. With negative composite drops the reverse takes place. From this one would be inclined to conclude that the two coexisting coacervates always have the same charge sign. From this it would again follow that the line of reverse of charge in the triangle connects two coexisting coacervates. But this reasoning is inadmissible, as the direction of movement of the enclosed coacervate drop is no indication of its charge sign. For any enclosure (vacuole, carbon particle, oil drop) moves in this way in a coacervate drop. Yet the theory that the line of reverse of charge connects two coexisting coacervates is plausible.

Suppose the connecting lines of the coexisting coacervates have a different course, for instance the one in Fig. 3b.

Then mixtures of the colloid compositions 2, 3 and 4 break up into coexisting coacervates of colloid compositions 1 and 5. One of these two is then the enveloping coacervate of the composite drops, and must therefore always give the same charge sign to these drops. But this is contradictory to our experience, for the composite drops formed from 3 are uncharged, from 2 they are positive and from 4 they are negative (see previous section).

In the same way any other direction of the connecting lines is contradictory to our experience, unless the line of reverse of charge itself is a connecting line of coexisting coacervates (Fig. 3c). What has been said of the line of reverse of charge also applies practically to the other lines of constant electrophoresis velocity, whose course follows from the data of the table in § 4. Therefore they are drawn in full in Fig. 2 for so

¹⁾ Dyes stain the coacervate of high N percentage many times more intensively than the coacervate of high A percentage and thus the former can at once be recognized microscopically.

far as their course falls within the area of the coexisting coacervates. This indicates that they may be considered as approximately connecting lines of coexisting coacervates.

Two lines with constant U (electrophoresis velocity) will touch the closed curve of the two coexisting coacervates at the place of the two critical points.

While the location of the point of contact nearest to vertex G is practically known, the other one is not known on account of the absence of electrophoresis measurements. But from the coacervate volume curves of Fig. 3 we can obtain a control concerning the correctness of the location of the critical point mentioned first and indications concerning the location of the second critical point.

As an example we take the mixing series pictured in Fig. 1. Here from left to right on entering the area of the coexisting coacervates we note the presence of the coacervate layer of high A -percentage and we see that the layer of high N -percentage increases from zero upward. On leaving the area of the coexisting coacervates on the other hand we see that the coacervate layer of high N -percentage is present and that the layer of high A -percentage decreases to zero. For this reason we have added the letters (n) or (a) to the mixing percentages in the survey table of § 3, in order to indicate what coacervate is present in abundance on passing the boundary. At the critical points mentioned the curve branch of the coacervates of high A -percentage must pass into that of the coacervates of high N -percentage. From Fig. 4 in which we have indicated the coacervates of high A -percentage by open circles, those of high N -percentage by black dots (see survey Table 3), we see that the critical point on the side of vertex G of the triangle, as indicated by the course of the lines with U -constant, lies indeed between the series of the white points (left) and of the black points (right). Reversely the place of the other critical point is indicated by the space between the white and black dots on the other side of the area of the coexisting coacervates. Dotted lines within the area of the coexisting coacervates indicate the probable course of the connecting lines of the coexisting coacervates near this critical point.

Summary.

1. The occurrence of coexisting coacervates in mixtures of purified gelatine, Na-arabinate and Na-Yeast nucleinate was investigated in the presence of diluted buffers at pH ca. 3,7 and the results were put out in ternary diagrams.
2. The results agree very well with the results previously obtained with unpurified colloid preparations. The investigation was extended with electrophoretic measurements and with the measurement of coacervate volumes.
3. Thus the probable direction of the connecting lines of coexisting complex coacervates in the ternary diagram could be determined.

Leiden, Laboratory for Medical Chemistry.

Biochemistry. — *Behaviour of microscopic bodies consisting of biocolloid systems and suspended in an aqueous medium. VI. A. Auxiliary apparatus for studying the morphological changes of coacervate drops. B. Preparation and behaviour of composite drops consisting of coexisting complex coacervates.* By H. G. BUNGENBERG DE JONG. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of February 28, 1942.)

1. When studying coacervate drops we often felt the need of an apparatus in which a coacervated system could be kept in stock in which on the one hand the coacervate drops could coalesce to larger ones, while on the other hand these larger drops remain suspended in their medium for some length of time.

A solution of these requirements was formerly found in an apparatus we called by the name of "Kreiströhre"¹⁾. This apparatus consists of a circular tube connected by spokes to a central axis and which rotates slowly. The coacervated system only partly fills the tube, so that when rotating it is always flowing. Although this apparatus has proved very useful it also has certain disadvantages, it is namely impossible during rotation to take small samples from it in order to check any changes in the coacervate drops from moment to moment, nor is it possible to add substances to study their effect on the coacervate drops.

The apparatus pictured in Fig. 1 removes these difficulties. Here the coacervated

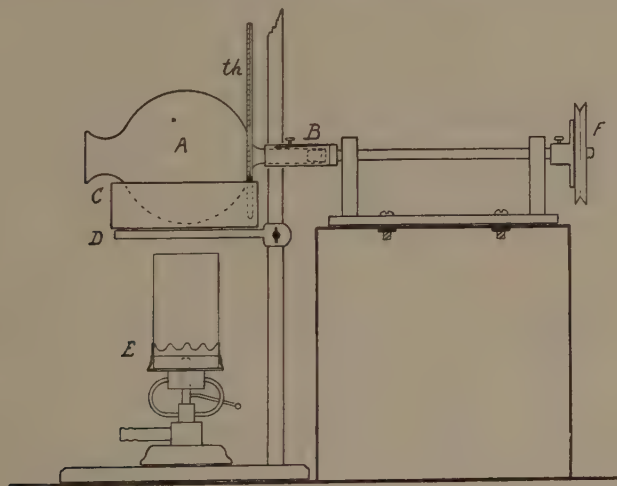


Fig. 1.

system is placed in a glass sphere (A), which turns horizontally round its axis. At the back this sphere is narrowed to a tube which is closed with a thin rubber stop and which fits into a copper case (B) of the horizontal axis, into which it is fixed by means of a screw.

In front there is a bell-shaped opening, through which the apparatus may be filled; during the rotation substances may be added and with a pipette or glass rod a sample may be taken from it for examination under the microscope. The sphere (A) is submerged

1) H. G. BUNGENBERG DE JONG and O. BANK, *Protoplasma* **33**, 322 (1939).

in a basin of water (C), the temperature of which can be read with the thermometer (Th), The basin is placed on a ringstand (D) and can be heated by means of an Argand burner (E). The rope puller (F) is connected with a "Saja" synchro motor by means of a metal spiral string, so that the sphere undergoes about 30 rotations per minute. This is a suitable rotation velocity for the apparatus used by us, in which the diameter of the glass sphere was ca. 10 cm. With the help of this apparatus we lately studied the composite drops consisting of two complex coacervates, formed in the system gelatine-gum arabic-Na-Nucleinate when the pH is brought at a suitable value¹⁾.

2. *pH section in which coexisting coacervates occur with colloid proportion gelatine : gum arabic : nucleinate = 3 : 1 : 1.*

Although in the system gelatine—gum arabic—Na-Nucleinate with constant pH there is a complete series of mixing proportions of the three colloids, with which coexisting complex coacervates are formed, these mixing proportions by no means all lend themselves to obtaining composite drops suitable for morphological studies. With many of these mixing proportions for instance, composite drops are obtained, with which in the coacervate of high gum arabic percentage a great number of smaller coacervate drops are enclosed of high nucleinate percentage, which coalesce with some difficulty only. For a morphological investigation it is desirable that the enclosed coacervate of high nucleinate percentage, with favourable pH values at least, does easily coalesce to one or to a few drops, so that the starting point is a simple morphological initial condition.

This requirement is fulfilled by the colloid mixing proportion gelatine : gum arabic : Na-Nucleinate = 3 : 1 : 1.

Here follow the results of coacervate volume and pH measurements at 40°²⁾ for this colloid proportion, namely on the one side the pH was varied by adding hydrochloric acid, on the other by acetate buffers. In both cases our starting point was a stock sol consisting of 3 g gelatine + 1 g gum arabic + 1 g Na-Nucleinate in 250 cc dist. water³⁾. With the first series the composition of the mixtures was 10 cc stock sol + a cc HCl 0,1041 N + (5 - x) cc H₂O. The coacervate volumes were determined in sedimentation tubes and noted down after one night in the thermostat, while pH measurements were taken of a number of fresh prepared mixtures (a = 0,25, pH = 4,86; a = 0,5, pH = 4,42; a = 0,7, pH = 4,10; a = 0,9, pH = 3,77; a = 1,1, pH = 3,43; a = 1,3, pH = 3,16; a = 1,5, pH = 2,85). The results are pictured in Fig. 2a. We see that in the pH section of ca 2,9—4,3 coexisting coacervates occur. The coacervate of greater specific gravity and of high nucleinate percentage (which however also contains a little gum arabic) is indicated in the figure as G + N + a, the coacervate of less specific gravity and of high arabinate percentage (but also containing some nucleinate) is indicated as G + A + n.

We also studied the morphological appearance in fresh prepared mixtures: it appeared that the most favourable picture was obtained as regards the coalescence of the G + N + a drops enclosed in the G + A + n drop to few larger drops in the centre of this pH section. We also examined on a heated object table the details of the desintegration phenomena in the electric field, and we found that with a = 0,9 the picture occurs that is characteristic of a negative complex coacervate, while with a = 1,1 the picture is indicative of a positively charged complex coacervate. The morphologically

¹⁾ H. G. BUNGENBERG DE JONG and A. DE HAAN. *Biochem. Zeitschr.* **263**, 33 (1933). H. G. BUNGENBERG DE JONG and E. G. HOSKAM. *Proc. Ned. Akad. v. Wetensch.*, Amsterdam, **45**, 387 (1942).

²⁾ Here we thank Miss E. G. HOSKAM for her assistance with these measurements.

³⁾ Colloid Preparations: Gelatine F00 extra of the "Lijm- en Gelatinefabriek 'Delft'" at Delft; gum arabic: gomme Senegal petite boule blanche I of ALLAND et ROBERT, Paris; Na-Nucleinum e faece of E. MERCK.

most favourable picture, therefore, occurs here near the point of reversal of charge.

With the second series we used acetate buffers with constant Na-Acetate concentration and varying acetic acid concentration¹⁾ according to the direction: 10 cc stock sol + 5 cc buffer.

The buffers to be used were prepared by placing each time 20 cc 0.1 N Na-Acetate in measure flasks, adding first b cc 2 N acetic acid and finally dist. water until 100 cc.

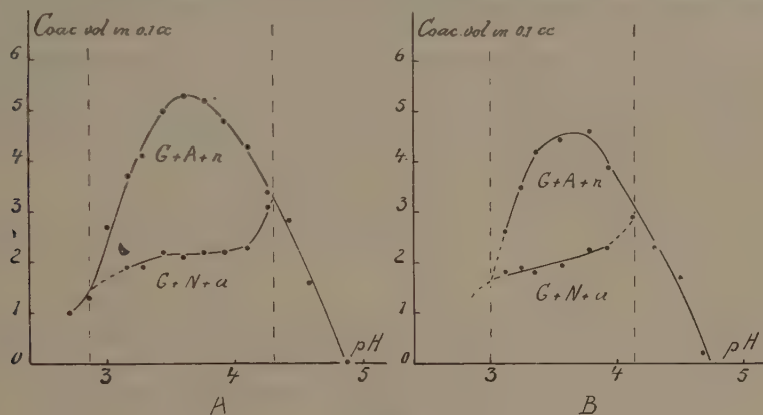


Fig. 2.

In order to ascertain in how far these diluted buffers will manage to enforce on the colloid mixture the pH of the buffer itself, we also prepared some mixtures of composition: 10 cc dist. water + 5 cc buffer, and its pH was also measured at 40°. The

TABLE 1.

b	pH of the blank series	pH of the coacerv. system	Coacervate vol. **) in 0.1 CC after one night ²⁾
1	4.64	4.83	—
1.5	—	4.67 *)	0.2
2.5	4.26	4.48	1.7
4	—	4.28 *)	2.3
6	3.90	4.12	? — 2.9
10	—	3.92 *)	2.3 — 3.9
15	3.55	3.78 [*)	2.2 — 4.6
25	—	3.56 *)	1.9 — 4.4
40	3.12	3.36	1.8 — 4.2
60	—	3.23	1.9 — 3.5
80	2.86	3.11	1.8 — 2.6

* = Interpolated.

** = Where there are two coexisting coacervates, the first figure indicates the volume of the coacervate of high nucleinate percentage, the second the total of the volumes of the two coacervates.

1) As neutral salts have a neutralizing effect on the complex coacervation, it is necessary when using buffers to keep the concentration of the buffer salt constant and preferably small.

2) The two coexisting coacervates sedimentate within a few hours to a turbid layer, but the separation into two layers (sometimes still turbid) is only accomplished after one night.

table shows that the pH of the colloid mixtures is 0.21–0.25 higher than that of the buffers. With the little concentration of the buffer salt given (6.7 m.aeq. p. L.) the pH difference cannot be expected to be very small. The results of this second series of measurements are given in Table 1 and Fig. 2b.

Also in the second series we find the most favourable morphological pictures in the direct surroundings of the maximum of the $G + A + n$ curve. Here ($b = 10$ and 15) the equilibrium liquids are also clearest.

When the two series are compared, the neutralizing effect of the buffer salt is evident. The $G + A + n$ coacervate is the least salt resistant and we see that in the series with buffers the maximal volume of the $G + A + n$ coacervate is slighter than in the series without buffers. The pH section in which the coexisting coacervates occur with buffers (ca. 3–4.1) is also narrower than in the series without buffers (ca. 2.9–4.3). The maximum of the $G + A + n$ curve has not with certainty shifted.

3. Preparation of drops of morphologically simple construction consisting of coexisting complexes coacervates.

We have already mentioned in 2 that the mixing proportion gelatine : gum arabic : nucleinate = 3 : 1 : 1 is eminently suitable for the preparation of coacervate drops of morphologically simple construction. We must here add some restrictions.

1. The success depends on the right pH, 2. there seems to be some variability in the Na-Yeast-nucleinate preparations: the contents of one bottle may be suitable for our purpose, whereas with those of another bottle the enclosed coacervate drops of high nucleinate percentage do not readily or not at all coalesce to one or few larger ones¹). Preliminary experiments show that in that case the addition of a little CaCl_2 (e.g. 5 m.aeq. final conc.) cause some improvement. We have not yet had an opportunity of determining the cause of this variability. There are indications that oxidation plays a rôle, for fresh samples are not suitable, but after standing during some weeks in contact with air they show the phenomenon.

In the following pages we shall ignore these complications, supposing that we are using a suitable nucleinate preparation.

In preparing the composite drops, then, the following method may be employed. We weigh 3 g gelatine + 1 g gum arabic (transparent pieces ground to a coarse powder) + 1 g Na-Nucleinate. The three powders are mixed and quickly poured into a 200 cc Erlenmeyer filled with 100 cc dist. water. It is at once closed with a rubber stop and vigorously shaken up and down for ca. 5 minutes. This prevents the gum arabic and especially the Na-Nucleinate from sticking together, which easily happens when the Erlenmeyer is not shaken. We now place the mixture in the refrigerator (ca. 6° C.) until the next morning, when we put the Erlenmeyer for at least 10 minutes in a waterbath of 60° C. under occasional shaking, after which our mixed stock sol is ready to be used. When one is likely to use it for several days the stock sol can be divided over several flasks, which are then kept on stock in the refrigerator. This is better than melting the entire stock each time on the waterbath, as changes occur in the stock sol when it is kept at 60° C. for some length of time, after which acidification does not produce coexisting coacervates²).

After heating the waterbath in which the sphere of the auxiliary apparatus (Fig. 1) rotates to ca. 50° 10 cc dist. water + 10 cc of an acetate buffer are pipetted into it.

¹) We have also seen that sometimes rod-shaped objects are formed owing to coacervate drops of alternately high arabinose and nucleinate percentage adhering together.

²) Probably owing to the fact that the nucleinate is then slowly hydrolyzed, for the property to form coexisting coacervates, which has been lost after one night at 60°, returned after the addition of a little nucleinate sol, not after the addition of gelatine or gum arabic sol.

the buffer having been prepared from 100 cc Na-Acetate 0,1 N + 100 cc acetic acid 1 N + 800 cc dist. water.

After a few minutes 5 cc stock sol is added. Within 5 minutes the small coacervate drops which form at once have coalesced to larger drops to such an extent that it is possible to continue our study of the effect of other additions. In order to observe it we take one drop from the coacervated system with a glass stick and place it on a starched object glass, which lies on the heated objecttable of the microscope. Thus viewed the composite coacervate drops are seen to consist of a homogeneous coacervate wall (the complex coacervate of high arabinate percentage) and embedded in it one or a few slightly vacuolized coacervate drops of the complex coacervate of high nucleinate percentage. (Compare the left pictures of rows A and B of Fig. 3).

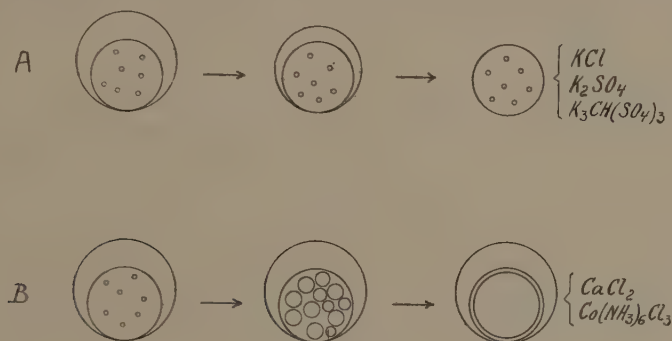


Fig. 3.

For the study of morphological changes of the composite drops it is convenient to be able always to distinguish the two coexisting coacervates. For this purpose we can recommend staining with toluidin blue (for our motives for this choice see § 9). We add 1 to 2 cc of a 0,04 % water solution to the contents of the sphere (25 cc coacervated system) of the auxiliary apparatus. The coacervate of high nucleinate percentage enclosed now takes an intensive green colour in the yellow light of the microscopization-lamp, the enveloping coacervate of high arabinate percentage is only slightly stained green.

4. Behaviour of the composite drops with respect to added salts.

Here follows a short discussion of the effect of salts when they are gradually added in solution to the coacervated system. It is seen that KCl , K_2SO_4 and $K_3CH(SO_3)_3$ cause the shell of high arabinate percentage to disappear, namely in the order $1-1 > 1-2 > 1-3$ with increasingly smaller concentrations. With $CaCl_2$ and $Co(NH_3)_6Cl_3$, the wall persists at lower concentrations, but vacuolization of the "nucleus" of high nucleinate percentage occurs and finally it passes into a hollow sphere and that in concentrations decreasing in the order $2-1 > 3-1$. These very interesting phenomena (compare Fig. 3) are being studied and we hope in due course to return to them.

5. Behaviour of the composite coacervate drops with respect to dyes.

In a previous publication¹⁾ we have described how methylgreen in small concentrations stains the enclosed coacervate of high nucleinate percentage much more intensively than the enveloping coacervate of high arabinate percentage. Other *basic dyes* (e.g. toluidin blue, pyronin, safranin, methylenegreen, methylene blue, Nile blue, neutral violet, cresylviolet, brilliant cresyl blue, trypanflavine) behave in the same way, although the contrast varies, being much smaller in some (fuchsin, crystal violet methylviolet).

¹⁾ H. G. BUNGENBERG DE JONG and A. DE HAAN, loc. cit.

One would be inclined to attribute the difference in stainability with basic dyes exclusively to stronger binding of the dye cation to the nucleinate colloid anion than to the arabinatate colloid anion. Contradictory to this theory is the fact that the same difference in stainability also occurs with respect to acid dyes. Eosin, erythrosin, ponceau red, indigo carmine and orange g also stain the coacervate drop of high nucleinate percentage intensively, and the enveloping drop of high arabinatate percentage weakly, although the positive colloid component in the two coacervates is the same (gelatine). We even find stronger stainability with dyes occurring as amphotones: methylred, rhodamin B and with colloid dyes: congorubin. With respect to the latter dye we note the following: when the red solution is added to the acid medium it is changed to blue. Gradually the staining of the composite coacervate drops is brought about, the enclosed coacervate of high nucleinate percentage being stained intensely red, the enveloping coacervate weakly red.

6. Localization of foreign particles taken up by the composite coacervate drops.

In general coacervate drops have the property of taking up particles presented in the medium, at least of collecting them at their surface. It was interesting to see how the composite coacervate drops behave in such cases.

When to the 10 cc of the buffer we add 10 cc dist. water + 1 cc 1% alcoholic solution of HgJ_2 , waiting a short time till the HgJ_2 has formed sufficiently coarse granules, after which we add 5 cc stock sol, the HgJ_2 appears to have been taken up in the composite drop and it has localized on the separation plane of the two coexisting coacervates¹⁾. Also with the other substances examined thus far, MnO_2 , carbon (norit), PbCrO_4 ²⁾, Ca oxalate, we find the same localization (compare Fig. 4a). Especially the

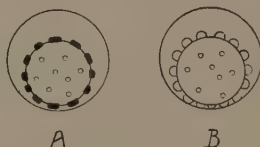


Fig. 4.

latter substance lends itself easily to demonstration as the gum arabic contains Ca, so that for coacervation we need only add a little Na-Oxalate to the stock sol³⁾.

7. Vacuolization phenomena on gelatinization of the composite coacervate drops in their own medium.

When a drop of the coacervated system is placed on a cold unstarched objectglass (on the non-heated object-table) the same picture as in 3. is at first observed. For the composite coacervate drops moisten the glass surface rather slowly and before this becomes

1) Care should be taken that the alcohol final concentration is small, as alcohol opposes the formation of coexisting coacervates. The same property but in a higher degree is found in acetone (with 10% acetone for instance, coexisting coacervates are no longer formed).

2) By double conversion of Pb-Acetate and K_2CrO_4 in the buffer. But this is no favourable object, as apparently secondary effects of the chromate act on the colloids (gelatine?), owing to which after a short time the coacervate drops coalesce to a sticky mass, which is deposited on the glass wall of the sphere.

3) To 40 cc stock sol should be added: 1 cc of a 2% Na-Oxalate solution, after which 5 cc of the turbid stock sol should be added as described above to 10 cc buffer + 10 cc H_2O .

troublesome the coacervate drops are already gelatinizing. In course of time two changes occur, the one consisting in the formation from the original equilibrium liquid of new, smaller, coacervate drops, resp. on further cooling in flocculation. The other change is concerned with the vacuolization phenomena of the large composite coacervate drops. On the one hand the vacuolization condition of the enclosed coacervate drop does not increase to any considerable extent, on the other hand new vacuoles are formed in the coacervate shell which originally was not vacuolized, while the localization is very striking, forming as they do a wreath on the boundary plane of the enclosed coacervate drop with the enveloping coacervate shell (compare Fig. 4b).

8. Gelatinized objects obtained by pouring the coacervated system into cold water.

When we pour 1 volume of the coacervated system into 10 volumes dist. water at room temperature, rapid gelatinization takes place. The objects thus obtained are distinguished from the objects which were formed by slow gelatinization in their own medium (see § 7) by a clearly visible structure of the wall of high arabinat percentage.

In consequence of fine vacuolization it shows granular spots, while the vacuolization of the "nucleus" is also much more accentuated. The cause of all this is the reduction of the salt concentration owing to the dilution of the system with dist. water. Thus the structures become even more noticeable when the gelatinized objects are washed with diluted acetic acid of ca. the same pH (0,001 N acetic acid, pH ca. 3,9). When afterwards salts are added to this 0,001 N acetic acid, the visibility of the structure decreases again, most markedly in the outer shell of high arabinat percentage. This may already look practically homogeneous while the "nucleus" is still strongly vacuolized. With higher concentrations the structure visibility of the nucleus also decreases. The phenomena described may be understood when it is remembered that the gelatinized objects are still complex systems (complex gels) ¹⁾.

9. Behaviour of the objects obtained according to 8 with respect to dyes.

With the gelatinized objects described above it is also possible to study the effect of salts on the staining process. It is then seen that in the washed condition (0,001 N acetic acid) not only the "nucleus" but also the shell of high arabinat percentage is stained (though to a less extent) by all sorts of dyes (basic, acid, etc.), but that the addition of salts decreases the stainability, most of the shell of high arabinat percentage, whereas the "nucleus" may be evidently stainable with much higher concentration.

Very interesting is the staining that takes place in washed objects with toluidin blue. The "nucleus" is again stained green, while the shell of high arabinat percentage is stained purple. Here we have a striking instance of *metachromasy* ²⁾. These two colours also occur when a little toluidin blue is added to a nucleinate sol (green) resp. to an arabinat sol (purple).

When salts are added it is seen already with comparatively small concentrations that first the purple staining of the arabinat shell disappears, to make place for a very weak green colour, while the "nucleus" retains its strong green colour. The explanation is that "arabinat staining" is already neutralized with small salt concentrations, but as the shell ($G + A + n$) also contains a little nucleinate, the green staining characteristic of this persists, naturally this staining is only weak compared with the "nucleus" whose nucleinate percentage is much higher.

As apparently in the case of toluidin blue in the presence of salts there is the favourable condition that it stains only in so far as there is nucleinate present, we have given the preference to this dye when choosing a stain for the composite coacervate drops in § 3.

1) H. G. BUNGENBERG DE JONG and O. BANK, loc. cit.

2) H. G. BUNGENBERG DE JONG and O. BANK, *Protoplasma* **32**, 489 (1939).

Another object was that visually indications may also be obtained as to the increase or decrease caused by variables of the partial mixability of the two coexisting coacervates. Increase (decrease) of it will become manifest as decrease (increase) of the contrast in intensity of the green colours of the two coacervates.

Summary.

1. An auxiliary apparatus for the study of morphological changes of coacervate drops is described.
2. We determined the pH section in which coexisting coacervates occur with colloid proportion gelatine : gum arabic : nucleinate = 3 : 1 : 1.
3. The composite drops of simple construction formed under favourable conditions, consisting of the coacervates mentioned in 2. were further studied with respect to dyes, salts, foreign particles and cooling.
4. Dyes effect the coacervate of high nuclein percentage far more than the coacervate of high arabinat percentage. Toluidin blue causes metachromasy.
5. KCl, K₂SO₄ and K₃CH(SO₃)₃ neutralize the coacervate shell of high arabinat percentage on increasing by smaller concentrations, on the other hand CaCl₂ and with smaller concentrations Co(NH₃)₆Cl₃ cause strong vacuolization of the enclosed coacervate drop of high nucleinate percentage, until finally it becomes a hollow sphere with a fairly thick wall.
6. Foreign particles are taken up by the composite coacervate drops and localized on the separation plane of the two coacervates.
7. On slow cooling in their own medium vacuoles are formed in the coacervate of high arabinat percentage; they form a wreath round the coacervate of high nucleinate percentage.
8. The behaviour of gelatinized objects obtained by pouring the coacervate into cold water with respect to salts and stains is discussed in detail.

Leiden, Laboratory for Medical Chemistry.

Biochemistry. — *Specific influence of cations on the water percentage of phosphatide coacervates.* By H. G. BUNGENBERG DE JONG and G. G. P. SAUBERT. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of February 28, 1942.)

1. Introduction.

The phosphatide trade preparations are to be considered as mixtures of phosphatides, phosphatidic acids and impurities (e.g. fats, oils etc.).

Each of these three classes plays its part in the colloid-chemical behaviour of these preparations. The phosphatidic acids are strongly bound to the phosphatides by LONDEN-V. D. WAALS and electrostatic forces and therefore their separation is not brought about by solvents (e.g. solution in aether and precipitation with acetone). Fats etc. are bound by LONDEN-V. D. WAALS forces to a less extent so that solvents can bring about more or less complete separation from the phosphatide-phosphatidic acid mixture. This process, however, causes a great change in the colloid chemical behaviour. Whereas sols of the original preparation flocculate (resp. coacervate) with salts (e.g. CaCl_2 , NaCl , etc.), this does not happen to the sols of the preparation purified with aether-acetone. When for the sol preparation fats, fatty acids etc. are added, these sols recover this property: they are "sensitized". The impurities present in the preparation such as fats etc., therefore play the part of *sensibilizers*. The part of the phosphatidic acids is an entirely different one. Whereas the perfectly pure phosphatide (e.g. Egglecithine) has in i.e.p. which lies close to the neutral point, the i.e.p. shifts to considerably lower pH values by a slight percentage of phosphatidic acid. On this account phosphatidic acids give a pronounced "acidoid" character to the phosphatide preparation. It is owing to their presence that especially the cation of the salt is all important for the behaviour of phosphatide preparations with respect to salts.

In sensitized sols (e.g. of unpurified preparations, resp. of purified preparations to which a known sensibilizer has been added) the effect of the cations may be studied in connection with the flocculation resp. coacervation phenomena. For each salt there is flocculation resp. coacervation in a certain section of concentrations. It was seen that variations are evident among the cations, in which not only the valency of the cation becomes manifest, but in which there also occur marked specific variations between cations of the same valency. To most of the phosphatides examined the following series applies, in which the concentration of optimal flocculation resp. coacervation increases from left to right:



The same series occurs when electrophoretically (with quartz particles suspended in them) the concentration is determined with which reversal of charge takes place (from negative to positive) of sols of purified phosphatide preparations¹⁾. So this concentration is low for Ca and increases in the series mentioned from left to right.

This cation series often occurs in physiological experiments (e.g. concerning the effect of salts on permeability), so that the presumption seems warranted that systems of a phosphatide + phosphatidic acid character take part in the protoplasmic membrane. But there are also indications that cholesterol (sterines) has a densifying effect on the plasmic membrane. It has appeared that cholesterol acts as a strong sensibilizer on purified phosphatide preparation, so that the supposition is warranted that the same three classes of substances: phosphatide + phosphatidic acid + sensibilizer are intricate parts of the plasmic membrane as are found in the usual phosphatide preparations of

1) H. G. BUNGENBERG DE JONG and P. H. TEUNISSEN, *Kolloid Beihefte* 48, 33 (1938).

trade. Elsewhere — on the ground of experiments — the theory of flocculation with salts of these sensibilized sols has been worked out (autocomplex flocculation, resp. auto-complex coacervation) ¹⁾, although at present we prefer a slightly different formulation of the systems formed (tricomplex systems). From this theory it may be foreseen that under comparable circumstances (compared each time at the optimal salt concentration, i.e. at the first approximation of the electrophoretic point of reversal of charge) the water percentage of these systems must also increase from left to right in the order of the cation series mentioned. Further, that at a constant and too large CaCl_2 concentration the water percentage must increase on the addition of NaCl .

These two points, which play a fundamental part in the theory of the protoplasmic membrane as autocomplex (tricomplex) phosphatide system, will be further investigated in the following pages.

2. Methods.

One of us has worked out a method of preparing phosphatide sols, which with salts produce sufficiently liquid coacervates at room temperature. On account of this they are suitable for comparing the mutual effect of salts with the aid of the determination of the coacervate volume. This method is actually a partial desensibilization of the original phosphatide preparation: 20 g "planticin alcohol solvable 90—95 %" of RIEDEL-DE HAËN is shaken with 200 cc 96 % alcohol at room temperature, when ca. 85 % being dissolved. This solution is poured into an Erlenmeyer of 200 cc which is placed in a thermosflask filled with 1 l water of 5°. After six hours a certain fraction has separated and deposited against the walls and on the bottom. This fraction is sensibilized to a greater extent than the phosphatide remaining in solution. The remaining clear solution is poured out in a thin jet into 800 cc dist. water under constant mixing and the sol obtained is liberated from alcohol by dialysts during 3 to 4 days (Sterndialysator; dialysis at 6°). The concentration of sols obtained in this way is of the order of 1 %. It is now seen that a certain subsequent heating treatment is necessary for the sol to produce sufficiently liquid coacervates at room temperature (e.g. heating for $\frac{1}{2}$ hour to 90°, resp. 24 hours to 40°).

The importance of coacervability at room temperature is in the possibility rapidly to execute more extensive series of experiments. For then it is possible after coacervation to centrifuge the sedimentation tubes. This can namely only be done when the centrifuging is done at the same temperature as the coacervation, as the temperature has a very great influence on the water percentage of the phosphatide coacervate and hence on the coacervate volume.

In a series of flasks we made mixtures of the composition: a cc salt solution + (20—a) cc dist. water, adding 5 cc phosphatide sol to each mixture. After shaking each time 5 cc from each mixture is pipetted into two sedimentation tubes. The tubes are placed into the hollows of a wooden block. Four of these blocks, each with 6 sedimentation tubes are then placed in the holders of a large "Ecco" centrifuge, so that 24 tubes can be centrifuged simultaneously (20 min. at 2000 rotations per minute) ²⁾.

Results.

A. Reversal of charge concentration of sol IV A.

Of mixtures of the composition indicated in 2, electrophoretic measurements were made with some salts. The results are given in the following table:

¹⁾ H. G. BUNGENBERG DE JONG und R. F. WESTERKAMP, *Bioch. Z.* **248**, 131, 309, 335 (1932).

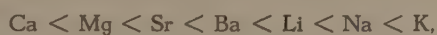
²⁾ For further description see G. G. P. SAUBERT, The influence of alcohols on the protoplasmic membrane and colloid models. *Recueil des Travaux Botaniques Néerlandais* XXXIV, 710 (1937) compare p. 733—755.

Log. C salt C in aeq. p.l.	Electrophoretic velocity			In arbitrarily selected units.
	Ca Cl ₂	Mg Cl ₂	Ba Cl ₂	Li Cl
0.60 — 2	— 63	— 113		
0.78 — 2	— 16	— 42	— 108	
0.90 — 2	+ 18			
0.00 — 1	+ 46		— 50	
0.08 — 1		+ 42		
0.20 — 1		+ 51	— 3	
0.30 — 1			+ 11	
0.60 — 1				— 135
0.78 — 1				— 61
0.00				+ 7
0.18				+ 46
Reversal of charge at log C =	0.83 — 2	0.93 — 2	0.02 — 1	0.97 — 1

So we see that the reversal of charge concentration from left to right increases in the order:



So these determinations were only made for the sake of control, that the order



which we have repeatedly determined for this type of phosphatides (in sensitized as well as in desensitized preparations) is not changed by the new method of sol preparation (partial desensitization).

B. *Comparison of the coacervate volumes with each other after coacervation of sol IV A with CaCl₂ MgCl₂ SrCl₂ BaCl₂.*

In the following table we give the results (each figure being the average of 2 duplicate determinations, differing no more than 0.2) of coacervate volume measurements with CaCl₂, MgCl₂, SrCl₂ and BaCl₂, in sections of the salt concentration round about the points of reversal of charge.

Coacervate volumes (in 0.01 cc)

Conc. m. aeq. p.l.	CaCl ₂	Mg Cl ₂	Sr Cl ₂	Ba Cl ₂
20	6.0	7.8	—	—
40	5.7	7.2	10.6	11.5
70	5.6	7.0	—	—
80	—	—	9.2	9.5
120	—	—	9.2	9.4
150	5.9	6.8	—	—
160	—	—	9.3	9.7
200	5.8	7.0	—	10.0
300	5.8	7.1	—	—
400	—	—	—	11.4

Theoretically it is to be expected that at or near the point of reversal of charge the water percentage of the coacervate is minimal; further, that at these minima the water percentage will increase from left to right in the order $\text{Ca} < \text{Mg} < \text{Sr} < \text{Ba}$.

As on the coacervation of phosphatide sols these variations of the water percentage are at once reflected in the changes of the coacervate volume:

- A. the coacervate volume curves must be curves with a minimum,
- B. these minima are expected near the reversal of charge concentrations.

This character of the curves is especially evident in BaCl_2 and SrCl_2 , but the minimum is much less pronounced in Mg and Ca and the curve branches which rise more suddenly are here outside the section of salt concentrations examined. It is further to be expected that the coacervate volume will increase from left to right in the order:

$$\text{Ca} < \text{Mg} < \text{Sr} < \text{Ba},$$

which was indeed found experimentally (see also Fig. 1).

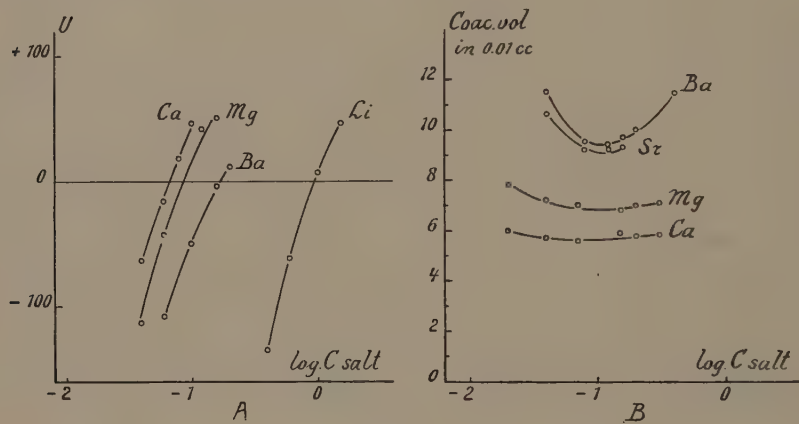


Fig 1.

C. Comparison of Ca, Mg, Cr, Ba, Li and N with another sol.

Analogous experiments were made with another sol, only with this difference that in order to obtain larger coacervate volumes, not 5 cc but 10 cc sol was present in the final volume of 25 cc. The results are given in the following table: (See p. 405).

Here we see that although in this series we used twice as much sol, the coacervate volumes are less than twice as large. Compare for instance in the previous table the values for CaCl_2 (there averagely 5.8 here 7.0 instead of 11.6). This indicates that the sol used here is sensibilized to a greater extent than the previous one. In agreement with this is the fact that the minimal character of the curves is even less marked here. As is seen from the table the only indication of this is the decrease of the coacervate volume with increasing NaCl concentrations.

Unfortunately our technique did not allow of the investigation of higher NaCl concentrations, as with 1040 m.aeq. and higher the specific gravity of the coacervates was less than that of the NaCl-solution, so that they came up on top, instead of being deposited as a layer in the calibrated narrow tube at the lower end of the sedimentation tubes. Hence the expected increase of the coacervate volume with higher NaCl-concentration could not be measured.

As for all the other salts the coacervate volume depends comparatively little on the salt concentration we have taken the average of the coacervate volumes, in order to compare the specific effect of the cations with each other (compare the lowest horizontal line of the table).

So here we do indeed see the expected order of the coacervate volumes:

$$\text{Ca} < \text{Mg} < \text{Sr} < \text{Ba} < \text{Li} < \text{Na},$$

Coacervate volumes in 0.01 cc.

Conc. m. aeq. p.l.	Ca	Mg	Sr	Ba	Li	Na
20	7.4	—	—	—	—	—
40	—	8.0	8.6	8.8	—	—
50	7.0	—	—	—	—	—
70	—	7.8	—	—	—	—
80	7.0	—	8.6	8.6	—	—
100	6.9	7.9	—	—	—	—
120	—	—	8.5	—	—	—
140	—	—	—	8.9	—	—
150	6.5 ¹⁾	7.9	—	—	—	—
160	6.9	—	8.7	—	—	—
200	7.1	8.0	—	9.0	—	—
240	—	—	8.9	—	—	—
300	7.0	7.9	—	—	—	—
320	—	—	8.8	—	—	—
400	—	—	—	9.4	10.0	—
480	—	—	—	—	—	13.3
600	—	—	—	—	9.7	—
640	—	—	—	—	—	12.9
800	—	—	—	—	9.9	12.7
1000	—	—	—	—	10.0	—
1040	—	—	—	—	—	— ²⁾
1200	—	—	—	—	10.0	— ²⁾
	7.04	7.92	8.68	8.94	9.92	12.97

indicating that the water percentage increases in this series, which is also the one of increasing reversal of charge concentrations (decreasing affinity of the cation for the phosphatide system).

With the glass electrode we measured the pH of a number of coacervated mixtures, which gave the following results:

CaCl ₂	20 m.aeq.	= 3.37	200 m.aeq.	= 3.37
MgCl ₂	40 "	= 3.37	150 "	= 3.34
SrCl ₂	40 "	= 3.47	160 "	= 3.45
BaCl ₂	40 "	= 3.44	400 "	= 3.37
LiCl ₂	400 "	= 3.38	1200 "	= 3.34
NaCl	480 "	= 3.35	1440 "	= 3.32

Although there are slight variations in pH it does not by any means follow that the salts themselves have a systematic effect on the pH of the coacervated systems. In this case therefore, the specific cation effect cannot be attributed to the consequences of primarily occasioned pH changes.

D. Antagonism CaCl₂ — NaCl.

In some series we measured the effect of increasing NaCl-concentrations with constant CaCl₂-concentration. The following is the result of the series with 20 m.aeq. CaCl₂.

¹⁾ This value, which is probably incorrect, has been left out of consideration in calculating the average.

²⁾ The specific gravity of the coacervates is less than that of the NaCl solution.

Effect of NaCl on the coacervate volume with 20 m. aeq. CaCl_2

Conc. NaCl in m. aeq p.l.	Coacervate volume in 0.01 cc.
—	3.35
20	3.3
50	3.6
120	4.0
400	4.6
560	4.4

We see here that as may be expected from the theory of the autocomplex systems, NaCl causes an increase of the water percentage (here = coacervate volume). But this influence will be the less evident, as the CaCl_2 concentration which is kept constant is chosen higher. With 80 resp. 160 m.aeq. CaCl_2 this can no longer be seen as a pronounced increase of the coacervate volume.

SUMMARY.

1. We measured the coacervate volumes of phosphatide sols coacervated with salts (chlorides), the order of increasing volume was found to be:



2. This order is the one of increasing reversal of charge concentration.
3. The theory of autocomplex coacervation foresees that in the order of increasing reversal of charge concentrations the water percentage of the coacervate will increase with optimal coacervation.

With phosphatide coacervates the coacervate volume is a measure for the water percentage of the coacervate and as moreover, the reversal of charge concentrations increase in the order:



the results of 1 may be fully expected.

4. With not too great CaCl_2 concentrations the coacervate volume increases with increasing NaCl concentration. This effect (increase of the water percentage) is also to be foreseen from the theory of autocomplex coacervation.

5. The significance of the foregoing for the problem of the nature of the protoplasmic membrane was touched upon.

Leiden, Laboratory for Medical Chemistry.

Anatomy. — *Biologic-anatomical Investigations on the Bipedal Gait and Upright Posture in Mammals, with Special Reference to a Little Goat, born without Forelegs. II.* By E. J. SLIJPER (Utrecht). (From the Institute of Veterinary Anatomy of the State University, Utrecht, Holland; Director Prof. Dr. G. KREDIET.)

(Communicated at the meeting of March 28, 1942.)

5. **Length of the ilium, m. gluteus medius.** HOWELL (25) and ELFTMAN (14) tried to demonstrate, that in bipedal Rodents and Marsupials the ilium was proportionally shorter than in their quadrupedal relatives. WATERMAN (64) on the contrary believes, that in upright going Primates the ilium is longer than in quadrupedal monkeys. These authors, however, used either the length of the whole ilium, or the length of the iliac blade as a fixed dimension to compare with the postsacral part of the ilium. For this postsacral part is the only part of the ilium, which is directly connected with the transmission of the body-weight to the supporting leg. My own researches surely showed that only the body-length may be used as a standard dimension, with which the dimensions of the pelvis may be compared.

The data given in table 3 show, that in all bipedal and upright going mammals, with the exception of man, the ilium has been lengthened. In most mammals this lengthening exclusively has been brought about by a lengthening of the presacral part of the ilium (the iliac blade). Only in hanging-climbing mammals the postsacral part too is a little elongated. It is further shown, that the length of the postsacral part of the ilium only to a very small extent depends on statical or mechanical forces. The length of this part is chiefly connected with the demands of space in the pelvis. Together with the length of the sacrum, the width of the lumbo-sacral and the width of the ilio-lumbar angle, the length of the postsacral part of the ilium determines the position of the pelvic inlet. The longer the sacrum and the narrower the ilio-lumbar angle are, the longer the postsacral part of the ilium must be, in order to bring the pelvic inlet in a plane that lies caudal to the last sacral vertebra (see for example *Capra hircus* L. and the Primates).

As we have seen above, in bipedal mammals the ilium has been elongated by an increase in length of its presacral part. This is easy to understand, because the length of the ilium determines the length of the fibres of the m. gluteus medius. In consequence it determines the width of the angle that the upright or semi-upright body can make with the horizontal plane. Hence in the series of climbing, bipedal jumping and hanging-climbing mammals, the length of the ilium and in consequence the length of the gluteal fibres increase gradually. But in man, whose body is perfectly upright and kept in balance on the lower extremities, the ilium is comparatively short and the m. gluteus medius shows a comparatively weak development. The broadening of the ala ilii is connected with the broadening of the whole body in anthropoids and man [SLIJPER (61)].

In the bipedal goat, which could not very easily attain an upright posture since it had no tail acting as a counterweight to the body, one might have expected, that the ilium would have been very long. Table 1, however, shows that this bone is nearly as long as in the control-animal. This may easily be understood since in the goat — as in most Ungulates — the length of the fibres of the m. gluteus medius only to a certain extent depends on the length of the ilium. In the greater part of the Ungulates the muscle originates not only from the ala ilii but also, by the so-called gluteal tongue, from the superficial aponeurosis of the m. longissimus dorsi in the lumbar region cranial to the iliac crest (fig. 4). This gluteal tongue is absent in *Proboscidea* [CUVIER (11), MIALL and GREENWOOD (40), EALES (13)], *Rhinocerotidae* [HAUGHTON (22)], *Camelidae* (own observations) and *Dicotyles tajacu* (L.) [CUVIER (11)]. The tongue is com-

paratively small in the pig and all Ruminants [see for example KOLESNIKOV (31) and REISER (51)], but it shows a large development in the *Equidae* and especially in the *Tapiridae* [MURIE (42), CUVIER (11)]. The tongue is absent in all other mammals, the bipedal mammals included [CUVIER (11; *Macropus*), PARSONS (48; *Pedetes*), HOWELL (25; bipedal Rodents)]. Only in the kangaroo-rat (*Dipodomys*) HOWELL (25) described a small gluteal tongue.

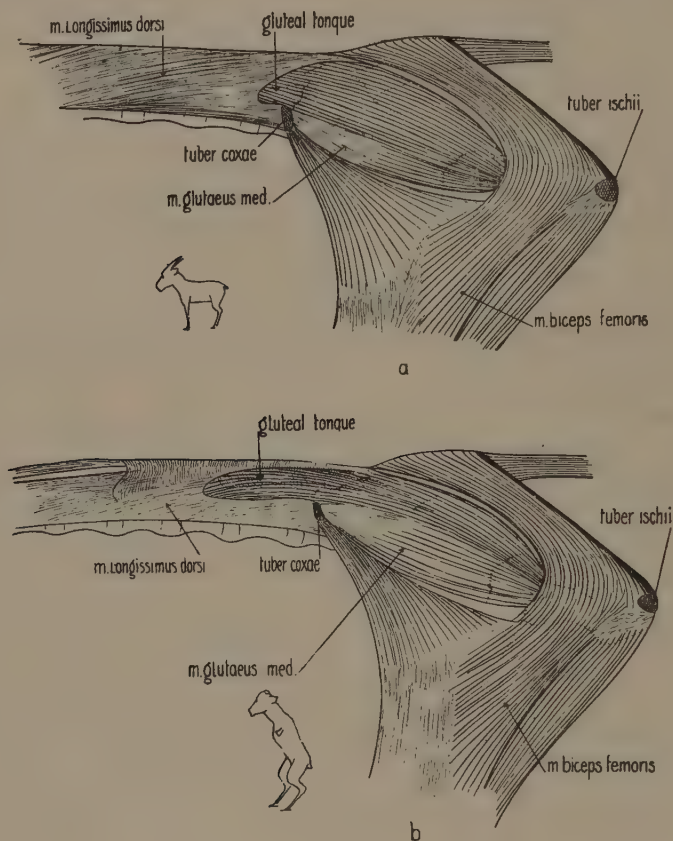


Fig. 4.

Lateral view (left side) on the muscles of the pelvic region in the normal (a) and the bipedal (b) goat. Special notice should be taken of the tongue of the *m. gluteus medius*.

In the normal quadrupedal goat the tongue had a length of 25 mm. In the bipedal animal it had a length of 50 mm; moreover it was much thicker and it originated not only in the normal way from the superficial aponeurosis of the longissimus dorsi by fleshy fibres, but also by a system of comparatively long and flat superficial tendons, which were attached to the aponeurosis and the fascia lumbo-dorsalis in the median line (fig. 4). FULD (17) and KOWESCHNIKOWA und KOTIKOWA (32) found in the bipedal dog and cat only an increase of weight of the *m. gluteus medius*. In the bipedal goat the *m. gluteus accessorius* too showed a better development than in the control animal.

6. Length of the ischium. The data of table 3 show, that in the series of walking, climbing, bipedal jumping, hanging-climbing mammals and man, there has taken place a very marked increase in length of the ischium. ALEZAIS (2) has already shown for the

kangaroo, that the lengthening of the ischium causes an increase in length of the hamstring-muscles and that it enables the adductor muscles to act as retractor muscles too. In consequence the lever-arm of the muscles that bring the body in an upright posture, is lengthened and the angle of erection widened. Moreover, the distance over which the femur can be moved when the animal jumps, is enlarged to a comparatively great extent. Thus the marked increase in length of the ischium of the bipedal goat (23%; table 1) should not cause a surprise.

7. Symphysis pelvis. As has already been shown sub 1, the factors determining the length of the symphysis pelvis, are the weight that is supported by the hindlegs, the power of the propulsive stroke of this leg, the position of the acetabulum with regard to the ilio-sacral joint, as well as the manner of locomotion of the animal. So it appeared from the large amount of data given by MIJSBERG (45), that in quadrupedal mammals the length of the symphysis especially depends on the absolute size of the animal and its manner of locomotion (jumping or not). In bipedal mammals the position of the acetabulum does not differ very much from that in their quadrupedal relatives; only in the anthropoid apes the pelvis is very broad at the acetabular joint, to give the animal a large supporting surface. In opposition to the conclusions of ELFTMAN (14) and WEIDENREICH (65), but in accordance with the data of MIJSBERG (45) and HOWELL (25), table 3 shows, that — with the exception of man (see sub 1) — in all bipedal and upright mammals there has taken place an increase in length of the symphysis pelvis.

In spite of the position of the acetabulum (see sub 8), the large weight supported by the pelvis and the unfavourable position of the body (see sub I), have caused in the bipedal goat an elongation of the pelvic symphysis (27%) and a marked thickening of the ischium and pubis (table 1, fig. 3).

8. Width of the pelvis. The data given in table 3 show, that in all bipedal and upright mammals the whole pelvis is wider than in allied quadrupedal animals. The widening of the pelvis enlarges the supporting surface of the hindlegs. This is especially striking in Primates [see also VAN DEN BROEK (7) and SLIJPER (61)]. A special divergence of the ischia or a convergence of the ilia does not occur in bipedal mammals.

Besides a small widening of the pelvic inlet, the bipedal goat on the contrary showed a very striking narrowing of the pelvis at the acetabulum and a compensating divergence of the ischia (table 1, fig. 5). Perhaps the pelvis of the bipedal cat, described by

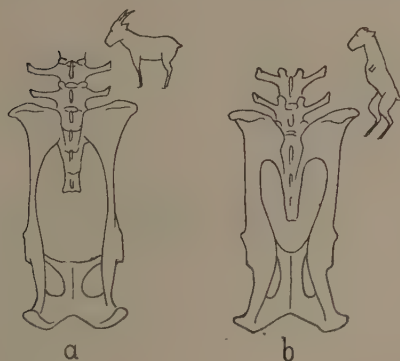


Fig. 5.
Dorsal view on the pelvis of the normal (a)
and the bipedal (b) goat.

KOWESCHNIKOWA und KOTIKOWA (32) showed the same characters. Most probably the width of the pelvis at the acetabulum decreased in the bipedal goat in order to diminish the exorotating force, which in this animal was extraordinarily large (unfavourable position of the body; no long tail). For if the acetabulum lies almost in the same paramedian plane as the ilio-sacral joint, at least one of the forces that cause the exorotation is considerably diminished. In relation to the width of the supporting surface the decrease

of the transverse diameter at the acetabulum partly is compensated by a lengthening of the collum femoris (see sub II and table 1).

9. Ligaments. In connection with the large body-weight that is transmitted to the ischium by the broad pelvic ligaments, it is not surprising at all, that in the bipedal goat these ligaments showed a very strong development.

10. Psoas musculature. On the whole the psoas musculature of the bipedal goat was apparently more feebly developed than in the normal one. The m. psoas maior originated only from the lumbar vertebrae (in the control-animal from the last thoracic vertebra too), the m. iliacus medialis originated only from the pelvis and the first sacral vertebra (in the control-animal from the last lumbar vertebra too), the m. psoas minor originated only from the centra of the 2d—last lumbar vertebra (in the control animal from the last thoracic until the last lumbar vertebra) and the area of insertion of this muscle at the pelvis was only half as large as in the quadrupedal goat. KOWESCHNIKOWA und KOTIKOWA (32) made the same observations in the bipedal cat, the weight of the m. iliopsoas amounted to only 87½ % from that of the quadrupedal animal.

The diminution of the psoas musculature may be explained by the fact, that in quadrupedal animals these muscles prevent the postsacral part of the pelvis from turning in a dorsal direction in consequence of the shock caused by the hindleg, when this comes down on the ground. In those bipedal animals that have no long tail the body-weight causes a rotation of the vertebral column in the ilio-sacral joint (see sub 1). This rotation neutralizes the dorsal rotation of the postsacral part of the pelvis. In bipedal animals its power is much larger than in quadrupedal ones, because the body-weight is not partly supported by the forelegs. In bipedal animals with a long and heavy tail, however, the body-weight is nearly counterbalanced by the weight of this tail. For that reason the psoas musculature of hanging-climbing mammals shows a comparatively feeble development [PRIEMEL (49)], while in bipedal Rodents and Marsupials especially the m. psoas minor is largely developed [PARSONS (47), SCHAPIRO (55), ELFTMAN (14)].

IV. Thorax.

In the normal quadrupedal land-mammals the shape of the thorax is characterised by: 1st. The fact that its walls are converging very markedly in a cranial direction; the thorax therefore has the shape of a bow-net. 2d. The fact that the proximal parts of the ribs are not, or at best to a very small degree, curved in a dorsal direction. 3d. The fact that in the middle of the thorax its transverse diameter is nearly as long as its vertical diameter. 4th. The fact that the lateral walls of the thorax converge very markedly in a ventral direction and that the sternum is very narrow. Among these quadrupedal land-mammals, however, two different types of the thorax again can be distinguished. The majority of the *Marsupialia*, the *Insectivora*, the smaller *Rodentia* and *Carnivora*, the *Prosimii* and the not-anthropoid *Simii* have an apertura thoracis that is more broad than high. The scapula of these animals is ventro-laterally directed (it makes an angle of average 45° with the vertical plane); the clavícula is long and dorso-laterally directed (see table 4 and fig. 6). In the bigger representatives of the above-mentioned orders and in general in the animals that show a more or less running type of locomotion [*Thylacynus*, *Cuniculus paca* (L.), *Leporidae*, majority of the *Carnivora* and all *Ungulata*; see fig. 1, 6, table 4 and SLIJPER (61)] the clavícula is very small or even wanting, the scapula shows a vertical position in the paramedian plane, or may even be ventro-medially directed, while the cranial part of the thorax is much more high than broad.

HASSE (20) made researches into the different shapes of the thorax in mammals. Previously I have already shown [SLIJPER (60)], that his denomination "kielförmig" better might be replaced by the name "reusenförmig" (shaped like a bow-net). Moreover from HASSE's considerations it does not appear very clearly to what causes these differences in the shape of the thorax must be ascribed. Most probably HASSE supposes

that the ribs are more or less deformed by the tension of the pectoral and serratus-musculature. It seems, however, better to accept, that everywhere in the thorax the bony substance of the thoracic wall arises in that direction in which it can resist the statical and mechanical forces in the best way. Moreover the shape of the thorax partially may be influenced by the fact, that the distribution of space in the thoracic cavity determines the position of the centre of gravity. So in running and especially in heavy mammals,

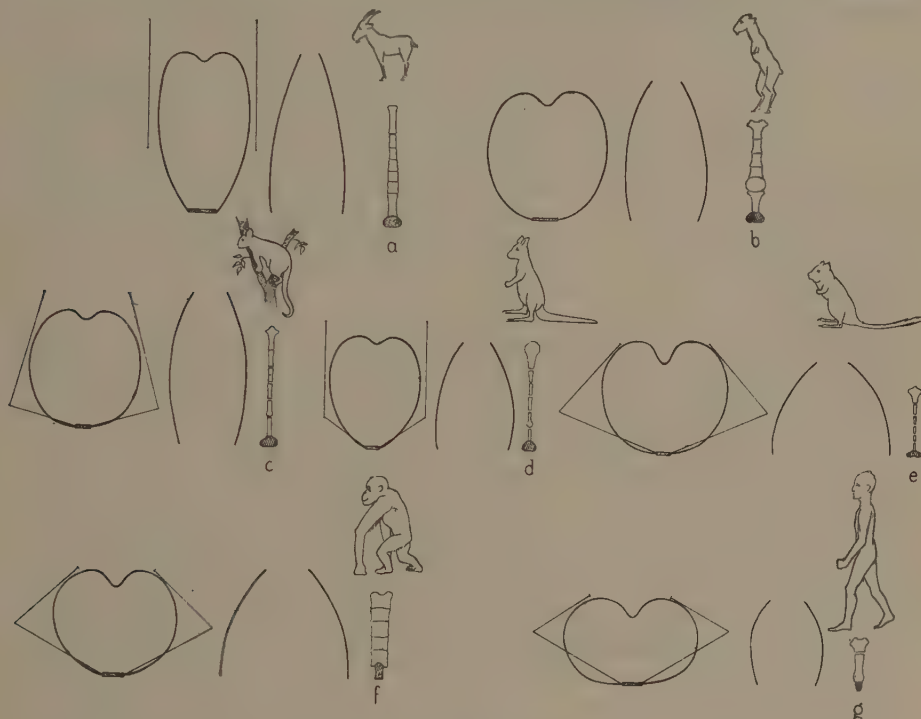


Fig. 6.

Schematic figures of the thorax of the different types of mammals, described in this paper. a. Normal goat (*Capra hircus* L.; running mammal). b. Bipedal goat. c. Cuscus (*Trichosurus vulpecula* (Kerr.); climbing Marsupial). d. Kangaroo (*Macropus giganteus* (Zimm.); bipedal, jumping Marsupial). e. Jumping-hare (*Pedetes caffer* (Pall.); bipedal jumping Rodent). f. Orang Utan (*Pongo pygmaeus* (Hoppius); hanging-climbing anthropoid ape). g. Man (*Homo sapiens* L.; bipedal walking Primate).

Every figure shows: Left: Transverse section of the cranial part of the thorax with the position of the scapula and the clavicle. Middle: Horizontal section of the thorax in the middle of the ribs. Right: Sternum from the ventral side.

the flat thorax enables the supporting foreleg to approach as nearly as possible the median plane (that is the plane in which the body-axis lies), while the nearly vertical ribs transmit the power from the body to the leg in the most favourable way. On the other hand, in these mammals the mobility of the foreleg has been considerably limited.

HASSE (20) thought that in every mammal whose body-weight is not supported by the forelegs, the thorax would acquire the shape of a barrel. Together with man and the anthropoid apes, however, he reckons among the mammals with a barrel-shaped thorax also the bipedal jumping Marsupials and Rodents as well as the aquatic mammals. I shall try, however, to demonstrate that this question is much more complicated than one can read from the work of HASSE.

In quadrupedal and especially in running mammals the body-axis has a horizontal position; the animals have an almost vertical scapula that lies very near to the body-axis and their upper arm forms part of the body (table 4, fig. 6). In climbing mammals the body-axis now and then is brought in a vertical position, the axis of the scapula as a rule is directed ventro-laterally and in many species the upper arm is almost perfectly free (see for example *Phascolarctos* and the Monkeys). All bipedal jumping mammals

TABLE 4. SOME CHARACTERS OF THE THORAX IN MAMMALS

TABLE 4. SOME CHARACTERS OF THE THORAX IN MAMMALS											
Species	Clavicula 1)	Position of scapula 2)	Greatest breadth in % of height		Dorsal curvature of ribs 3)	Number of ribs	Walls in % of body-length			Sternum breadth in % of length	
			Apertura thoracis	Middle of thorax			Dorsal	Ventral (sternum)	Sternum in % of dors.wall	1st rib	4th rib
<i>Equus caballus</i> L. (dom.)	-	v.	56	87	-	18	49	21	44	3	10
<i>Bos taurus</i> L. (dom.)	-	v.	50	71	-	13	48	21	44	3	25
<i>Lama glama</i> (L.)	-	v.	62	86	-	12	33	22	63	6	13
<i>CAPRA HIRCUS</i> L. CONTROL	-	v.	40	55	-	13	41	26	62	6	15
<i>CAPRA HIRCUS</i> L. BIPEDAL	-	v.	48	105	+	12	40	23	57	10	21
Average of running mammals	-	v.	52	75	-	14	43	22	53	5	16
<i>Sus scrofa</i> L. (dom.)	-	v.	70	85	-	14	48	20	40	2	19
<i>Canis familiaris</i> L.	-	v.	140	115	-	13	39	28	71	5	5
<i>Thylacinus cynocephalus</i> (Harris)	-	v.	100	66	-	13	44	27	62	11	8
<i>Lepus europaeus</i> Pall.	-	v.	100	87	-	12	41	27	69	5	5
Average of walking mammals	-	v.	102	88	-	13	43	25	60	6	9
<i>Trichosurus vulpecula</i> (Kerr.)	+	i.	140	93	-	13	50	27	55	10	9
<i>Phascolarctos cinereus</i> (Goldf.)	+	i.	150	121	+	11	45	24	52	31	11
<i>Sciurus vulgaris</i> L.	+	i.	153	108	+	12	45	33	75	17	7
<i>Anomalurus beecrofti</i> Fraser	+	i.-h.	500	120	+	15	40	20	50	30	10
<i>Cebus apella</i> (L.)	+	i.	110	88	+	12	41	29	70	17	8
<i>Trachypithecus pyrrhus</i> (Horsf.)	+	i.-h.	200	122	+	12	41	29	70	17	8
Average of climbing mammals	+	i.-h.	209	109	+	13	44	27	60	21	9
<i>Dendrolagus inustus</i> Müll.u.Schleg.	+	v.	180	84	+	13	45	31	70	14	7
<i>Bettongia lesueuri</i> Grayi Gould	+	v.	100	100	+	13	43	29	67	14	11
<i>Macropus giganteus</i> (Zimm.)	+	v.	100	100	+	13	46	28	60	18	18
<i>Pedetes caffer</i> (Pall.)	+	i.	600	140	++	11	35	20	57	34	10
<i>Jaculus jaculus</i> (L.)	+	i.	600	160	+	12	35	30	62	25	8
Average of bipedal jumping mammals	+	i.-v.	370	121	+	12	41	28	63	21	11
<i>Ateles paniscus</i> (L.)	+	h.	165	130	+	14	56	29	52	29	15
<i>Hylobates lar leuciscus</i> Geoffr.	+	h.	112	150	+	13	51	17	32	64	30
<i>Pongo pygmaeus</i> (Hoppius)	+	h.	131	119	+	11	63	26	41	38	34
Average of hanging-climbing mamm.	+	h.	136	133	+	13	56	24	42	44	26
<i>Homo sapiens</i> L.	+	h.	240	161	++	12	49	28	56	40	22
<i>Pteropus</i> spec.	+	h.	200	150	++						

1) + = Clavicula present. + = Clavicula very small. - = Clavicula wanting.

2) v. (h., i.) = Scapula has a vertical (horizontal, intermediate) position.

3) ++ = Proximal parts of ribs show a great curvature in dorsal direction.

+ = Curvature distinctly visible but not so high. + (-, -) = Proximal parts of ribs are directed laterally (ventro-laterally; ventrally).

- 1) + = Clavicula present. + = Clavicula very small. - = Clavicula wanting.
 2) v. (h., i.) = Scapula has a vertical (horizontal, intermediate) position.
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 + = Curvature distinctly visible but not so high. + (-, -) = Proximal parts of ribs are directed laterally (ventro-laterally; ventrally).

are characterized by a nearly free humerus and a body-axis that makes an angle of average 45° with the horizontal plane. The bipedal Marsupials have an almost vertical scapula lying close to the body-axis; the bipedal Rodents on the contrary have a latero-ventrally directed scapula. The relative shortening of the thorax in bipedal Rodents (table 4) must be ascribed to a lengthening of the lumbar region. In hanging-climbing mammals the position of the body-axis is almost vertical, the upper arm is perfectly free and shows a great mobility, the scapula is directed so much laterally that it has a nearly horizontal position. Quite naturally the above-mentioned characters of the apes are extremely developed in man. KNAUER (30), LOTH (34) and other authors have shown, that in hanging-climbing mammals and man the body is shortened. This shortening, however, principally bears upon the lumbar region [KEITH (29), SCHULTZ (57, 58), PRIEMEL (49), WILLIS (67)], so that the thorax proportionally is lengthened [SCHULTZ (58)]. Finally in the flying mammals the position of the body-axis is mostly vertical, the scapulae have a nearly horizontal position and the upper arm is quite free and very mobile.

In adaptation to the above-described characters of the body and the anterior extremity, the following changes of the thorax have taken place (table 4, fig. 6): The climbing

mammals only show an increase of the transversal and a decrease of the sagittal diameter of the apertura thoracis and the cranial part of the thorax. Moreover the first sternebra is much broader than in walking mammals. Besides the just mentioned characters, the bipedal Marsupials show a slight curvature of the proximal parts of their ribs in a dorsal direction. In the bipedal Rodents this curvature is more pronounced, the cranial part of the thorax is a little widened, the caudal part is very much widened and everywhere in the thorax the transverse section shows the beginning of a decrease of the sagittal and an increase of the transversal diameter. Finally in the hanging-climbing mammals and especially in man we meet a thorax with a very low sagittal and a very broad transverse diameter. The transverse section of this thorax has the typical oval shape, which is well known in man. The ribs show a pronounced curvature in the dorsal direction, by which the greater part of the space in the thorax is found at the dorsal side. The thoracic inlet and the cranial part of the thorax are much enlarged. In consequence of this enlargement the thorax has got the shape of a barrel, which HASSE (20) already described as the typical shape of the thorax of man. Finally the whole sternum is shortened and broadened to a very marked extent, while especially in older animals a synostosis of the different sternebrae has taken place.

KEITH (29) and RUGE (53) believed, that the broadening of the sternum and the synostosis of the sternebrae would be connected with the need for a greater area of origin for the pectoral muscles, especially because the sternum was so much shortened. This explanation, however, at the least is not quite satisfactory, because there is already a broadening of the first sternebra in mammals that have only a very feebly developed pectoral musculature (bipedal Rodents, Marsupials), while in *Choloepus*, which has an extremely strongly developed pectoral musculature, the sternum is very narrow. FREY (16) believes, that the broadening of the sternum would be a kind of compensation for the shortening of this bone. The broadening of the first sternebra, however, has taken place quite independently from the shortening of the sternum. In my opinion the shortening of the sternum is connected with the shifting of the space in the thorax in a dorsal direction. The broadening of the sternum in the first place seems to be connected with the broadening of the whole thorax, because the first sternebra is broadened as soon as a broadening of the thoracic inlet has taken place.

In his essential characters the thorax of the flying mammals (*Chiroptera*) quite agrees with that of the anthropoid apes and man. In almost every text-book of zoölogy one can read that the thorax of the aquatic mammals and especially that of the Cetacea has the same shape as the thorax of man and the flying mammals. Previously, however, I have already shown [SLIJPER (60)], that the thorax of the Cetacea has been influenced by quite other factors than that of the upright going land-mammals. In consequence the changes that have taken place in the cetacean thorax (widening of the whole thorax, special widening of the cranial part in adaptation to the torpedo-shaped body and to the stability, as well as a slight shifting of the space in a dorsal direction) differ very much from that of the upright going land-mammals.

From the foregoing description it is now evident that the shape of the thorax in bipedal and upright mammals is influenced principally by two factors. The first factor is the changed position of the body-axis. In connection with the stability of the body the upright posture demands a broadening of the body and a shifting of the centre of gravity in a dorsal direction, in order to bring this centre as near as possible to the body-axis [see also KEITH (29), RUGE (53), HASSE (20), BRAUS (6)]. The second factor is the position of the scapula and the upper arm. In a certain sense this must be considered as a limiting factor, because the broadening and widening of the cranial part of the thorax (which ultimately cause the barrel-shape of the thorax) can only take place, if the upper arm is completely free from the body and the scapula has an almost horizontal position.

The bipedal goat (table 1 and 4, fig. 6) showed the following characters: 1st. A very

marked increase of the transversal and a decrease of the sagittal diameter of the thorax. 2d. A curvature of the proximal parts of the ribs in a dorsal direction. 3d. A broadening of the apertura thoracis. Probably in connection with the shape of the neck, however, the typical ungulate shape of the aperture was present in the bipedal goat too. 4th. A widening of the cranial part of the thorax. 5th. A broadening of the whole sternum and a very slight (5%) shortening of this bone. Since one could not have expected, that in the time of a few months the thorax of this goat would have completely been changed into a human thorax, the above-mentioned changes may be considered as sufficient to confirm the considerations about the thorax of the bipedal and upright mammals. For example, this goat demonstrates very clearly that the broadening of the sternum cannot directly be connected with the demands of origin-area of the pectoral musculature. For in the bipedal goat the sternum is broadened in spite of the very feeble development of these muscles.

The dogs of FULD (17) did not show any change in the shape of their thorax. In the operated dogs of JACKSON (26), however, the thoracic index, which in normal dogs during the period of growth increases from 112 to 135, did not change at all during this period. In opposition to JACKSON, who expected too much of his dogs, I believe that his results are in perfect agreement with that of my own researches.

MARCUS (38) has shown, that in mammals the number of lobes of the lungs, among other factors (size and activity of the animal), can depend upon the shape of the thorax. According to MARCUS the widening and broadening of the thorax in Sirenia, Cetacea, anthropoid apes and man would have caused the decrease in number of the lobes of the lungs. These considerations are supported by the changes that have taken place in the right lung of the bipedal goat. In the control-animal this lung was composed of four lobes. In the bipedal animal there was only one lobus apicalis while the other lobus apicalis and the lobus cardiacus were coalesced with the lobus diaphragmaticus. The left lung on the contrary was quite normal. It is highly probable that these changes of the right lung were caused by the changes in shape of the thorax. For KATZ (28) recently has shown that already in the rhachitic and kyphotic thorax changes in the number of lobes of the lungs may very often occur.

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7. Die Kontakttheorie.

Wird in der somatischen oder psychischen Sphäre der Gleichgewichtszustand gestört, so entsteht reflektorisch das *Bedürfnis*, diese Störung aufzuheben. Das triebhafte Bedürfnis mobilisiert zu diesem Zwecke treibende Kräfte, die *Antriebe*, die direkt dem Ziel zusteuern. In der Zielstrebigkeit des Bedürfnisses und in der Zielgerichtetheit der Antriebe ist meistens auch das *Mittel* zur Erreichung des Zieles, der Weg zur Befriedigung des Bedürfnisses mit eingeschlossen. Dieser Vorgang lässt sich am deutlichsten in der reinen Triebphase beobachten, da hier der Prozess ohne Mitbeteiligung des Bewusstseins abläuft. Aber auch im geistigen Leben begegnen wir dieser im Biologischen wurzelnden gesetzmässigen Beziehung zwischen Bedürfnis und Antrieb einerseits, Bedürfnis und Mittel andererseits.

Entsteht bei einem Tier das Bedürfnis, seinen Hunger zu stillen, so wird es infolge angeborener triebhafter Mechanismen instande sein, dieses Bedürfnis durch zweckdienliche Mittel in geeigneter Weise zu befriedigen. Das Bedürfnis, welches nach Wiederherstellung des gestörten Gleichgewichtes strebt, veranlasst das Tier zwangsmässig zu motorischen Aktionen, die zu dem erstrebten Ziel führen.

Bei Tieren läuft der ganze Prozess — unseren gegenwärtigen tierpsychologischen Anschauungen nach — *unbewusst* ab. Der ganze Triebvorgang, vom Auftauchen des Bedürfnisses bis zur Ausführung der Tätigkeit, ist erbbiologisch präformiert; er kann als Ausfluss des einheitlichen und autonomen Triebmechanismus betrachtet werden. Alles läuft hierbei nach biologischen Gesetzen ab und stellt daher keine Anforderungen an das Individuum. Die ineinandergreifenden Phasen dieses Prozesses bestehen also erstens in dem Bedürfnis, das nach Befriedigung strebt, zweitens in dem Antrieb, der den Organismus in Bewegung setzt, und drittens in der instinktiven Regulierung des Triebs, mit deren Hilfe der Organismus Wege und Mittel sucht und meistens auch solche findet. Zielsetzung, Motivation und Entschluss, diese konstitutiven Elemente einer jeden bewussten Willenshandlung, fehlen hier gänzlich. Meine Anschauung lässt sich in folgendem Satz ausdrücken: asd Bedürfnis, ferner der Drang zur Befriedigung des Bedürfnisses und das Finden des zweckentsprechenden Mittels bilden eine *unzertrennliche biologische Einheit*. Von diesem Grundgedanken aus habe ich meine Trieb- und Instinktlehre aufgebaut.

Auch bei Menschen lassen sich solche unbewusste, in ihrem ganzen Verlauf durch Triebe determinierte Aktionen, die den Erregungszustand in einen Befriedigungszustand überleiten, feststellen. Im allgemeinen wird der Mensch freilich die Bedürfnisse und diejenige Objekte und Verhaltensweisen, die zur adäquaten Befriedigung seiner Bedürfnisse führen, erkennen und dafür sein Intellekt und seinen Willen einsetzen. Die Einschaltung des Bewusstseins hat zur Folge, dass gewisse rein triebhafte Bedürfnisse und die damit unzertrennlich verbundenen Mittel zur Befriedigung einen ganz anderen Charakter und eine andere Bedeutung erhalten. Die ursprünglichen Triebe sublimieren sich im Laufe der Menschheitsgeschichte; sie gehen in andere, teilweise in geistige Bedürfnisse über, ohne darum ihre ursprünglich triebhafte Natur gänzlich einzubüssen.

Die Erkenntnis der triebhaften Anlage der geistigen Bedürfnisse berechtigt uns, die vergeistigten Funktionen ihren ursprünglichen triebmässigen Formen gegenüberzustellen und entwicklungsgeschichtlich zueinander in Beziehung zu setzen und ihre zeitliche Aufeinanderfolge zu rekonstruieren.

Dies kann man auch bei der Sprache versuchen, bei einer Funktion, die ihresgleichen

in der Tierwelt nicht hat und dennoch zu gewissen primitiven Kontaktäusserungen eine entwicklungsgeschichtliche Beziehung aufweist. Die Berechtigung zu einem solchen Vorgehen gründet sich darauf, dass die Sprache und die primitiven, von Menschen und Tieren reichlich verwendeten Kommunikationsformen von demselben *Urprinzip* bestimmt und gerichtet werden. Obwohl die Sprache als eine spezifische Bildung des menschlichen Geistes zu betrachten ist, welche in keiner ihrer Funktionen mit irgendeiner tierischen Äusserung Verwandtschaft aufweist, ist es vom entwicklungspsychologischen und entwicklungsgeschichtlichen Standpunkt jedenfalls ein Gewinn, die menschliche Sprache mit primitiveren Kommunikationsformen durch ein *gemeinsames* Prinzip zu verknüpfen. Die Autonomie der Sprache wird dadurch nicht beeinträchtigt; es wird nur ein Faktor aufgezeigt, welcher zwischen animalischen und menschlichen Funktionen die Verbindung herstellt. Dieses Bindeglied ist das *Bedürfnis nach Kontakt*, in seiner höheren und spezifischen Form: *das Bedürfnis nach gegenseitiger Verständigung*.

Wenn wir auf die Verhaltensweisen der Tiere acht geben, so fällt uns auf, dass die Individuen einer grossen Anzahl von Tierarten das instinktive Bedürfnis haben, zueinander in mehr oder minder enge Beziehung zu treten. Von den niedersten Metazoen an bis hinauf zu den hochorganisierten Wirbeltieren ist der Trieb zum Beisammensein und Beieinanderbleiben anzutreffen. Dieser Bindung liegen in der Hauptsache zwei Grundtriebe zugrunde, nämlich die nach Selbst- und Arterhaltung. Diese Grundtriebe führen bei höheren Tieren zur Bildung von *Sozietäten*. Entweder sind die Sozietäten auf eine bestimmte Art beschränkt, wie die Schlaf-, Jagd- und Wandelgesellschaften, oder ihre Mitglieder gehören mehreren Arten, wie es bei den Brut- und Herdengesellschaften, den Parasiten und durch Adoption oder Symbiose zusammenlebenden tierischen Wesen der Fall ist. Unter den Tiergesellschaften sind bekanntlich die wichtigsten die sexuellen Sozietäten, deren höchste Form die Familie darstellt, bei der die Mitglieder unmittelbar durch den Geschlechtstrieb, mittelbar durch den verborgenen Zweck der Arterhaltung zusammengeschweisst sind ³⁵⁾.

Das Zustandekommen der Sozietäten im allgemeinen setzt bei den beteiligten Mitgliedern eine *Art sozialen Triebes* voraus, eine aktive Disposition zum Zusammenleben, die auf Bildung einer Gemeinschaft von beschränkter Dauer oder von beständiger gemeinsamen Lebensführung gerichtet ist. Wie dieser Trieb psychologisch gestaltet ist, wie weit er bei Tieren erkennbar wird, darauf wollen wir hier nicht näher eingehen. Eines ist jedenfalls sicher, dass dieser soziale Trieb, der zur Sicherung des Individuums, seines Nahrungserwerbs und Unterkommens dient, eine *kommunikative Tendenz* in sich schliesst. Ursprünglich führt diese Tendenz zu triebhaft reflektorischen Reaktionen; auf einer höheren Stufe gibt sie sich in mehr oder minder absichtlichen *Kontaktformen* kund. Von diesem sozialen Kontaktbedürfnis her entwickeln sich die verschiedenen Kommunikationsmittel gleichsam zwangsläufig, unter Berücksichtigung der angeborenen Anlagen sowie der Entwicklungsstufe der Art und der Einzelwesen.

Ist das Kontaktbedürfnis bei artgleichen Individuen gering, beschränkt sich das erstrebte Zusammenwirken bloss auf elementare Lebensbedingungen, so werden dementsprechend auch die Kommunikationsmittel einfach sein und einfach bleiben. Gestalten sich indessen die Lebensumstände komplizierter und ist das Individuum als solches (nicht nur als Mitglied der Sozietät) auf seine Artgenossen (bei domestizierten Tieren auf die Menschen) angewiesen, so entstehen Kontaktmittel, die infolge ihrer Mannigfaltigkeit und Differenziertheit ihre Ähnlichkeit mit den primitiven Ausdrucksweisen verlieren, obschon der Antrieb demselben Bedürfnis, derselben Triebquelle, entspringt.

Gehen wir bei unseren Ueberlegungen van der wohlbegründeten Annahme aus, dass die Laut- und Gebärdensprache ein Entwicklungsprodukt ist, welches aus minder entwickelten Kommunikationsformen dank zweckmässiger Anpassung entstanden ist, und versuchen wir von diesem Standpunkt aus auf Grund von unseren tier- und sprachpsychologischen Erfahrungen die zu der Sprache führenden Entwicklungsstufen zu rekon-

³⁵⁾ P. DEEGENER, Die Formen der Gesellschaftung im Tierreiche. Leipzig, 1918.

struieren, so bietet sich uns nur *ein* gangbarer Weg, nämlich der, welcher von den primitiven Kommunikationsformen ausgeht und mit logischer Konsequenz zu der Sprache vordringt³⁶⁾.

Die primitivste Kommunikationsform, die im ganzen Tierreich verbreitet ist und sich in reflektorischen Lautäusserungen und Bewegungen bzw. Haltungen manifestiert, können wir von unseren Betrachtungen ausschliessen, da sie vollkommen instinktiv vor sich gehen und bei ihnen die Kontakttendenz nicht mit Sicherheit festzustellen ist. Sie scheinen nichts anderes zu sein, als unmittelbare, reflektorische Reaktionen auf innere Vorgänge im Tierindividuum. Sie drücken somatisch bedingte emotionale Zustände aus; sie sind unmittelbare körperliche Folgerscheinungen von Lust- und Unlustzuständen, welche die Artgenossen reflektorisch zu gewissen zweckdienlichen Massnahmen veranlassen. So zielt der sog. Warnruf der Tiere keineswegs auf eine Kundgabe der Gefahr; er stellt vielmehr eine Schreckreaktion des individuellen Tieres dar, die zur Folge hat, dass Artgenossen, gegentllich auch artungleiche Tiere, die Flucht ergreifen. Dasselbe gilt auch für gewisse Lockrufe der Tiere, ferner für die Fühlerbewegungen der Ameisen, den Honigtanz der Bienen u.a. mehr. Diese rein biologisch fundierten, sozial zweckmässigen, vermutlich nicht-gerichteten Kommunikationsformen wollen wir aus der Betrachtung ausscheiden und erst einer höheren Kommunikationsform, den *adressierten Äusserungen* eine entwicklungspsychologische Bedeutung zusprechen, zumal hier das Kontaktbedürfnis in unverkennbarer Weise in Erscheinung tritt.

Die adressierten Lautäusserungen der Muttertiere setzen bereits einen *Kontakt* zwischen Individuen gleicher Art voraus. Dieser Kontakt ist zu charakterisieren als eine inter-individuelle Verbindung und als eine *von beiden Teilen* ausgehende Tendenz zum Zusammenwirken mit Hilfe von zweckmässigen und auf die Art abgestimmten Mitteln. Einer solchen Kontaktform bedient sich das Muttertier, um seine Jungen herbeizurufen, und auch das Männchen, um die Weibchen anzulocken. Das Verhalten der Tiere weist ausdrücklich auf die Existenz und Wirkung des Kontaktbedürfnisses hin. Absicht und Zielvorstellung lassen sich hierbei nicht annehmen. Es werden Instinkte mobilisiert, die kraft ihrer Eigennatur einen Kontakt zwischen den Beteiligten zustande bringen und zielstrebig wirken.

Trotz des emotional begründeten bilateralen Kontaktes ist der Weg von der adressierten Kontaktform zu der Sprache noch ein sehr weiter. Nun finden wir aber im Tierreich eine besondere psychisch fundierte Beziehung, die uns schon viel näher an die Sprachfunktion heranbringt. Wie wir bei der Darstellung der sog. Tiersprache ausgeführt haben, kommt zwischen Menschen und domestizierten Tieren gelegentlich ein Kontakt zustande, der schon viel mehr zu bedeuten hat als der instinktiv adressierte Ruf (Haltung). Öfters wird nämlich die Beobachtung gemacht, dass domestizierte Tiere gegenüber bestimmten Personen spontan ihrem Verlangen durch *Andeutung des erstrebten Zieles* Ausdruck geben. Hunde und Katzen z.B. tun ihr Verlangen, das Zimmer zu verlassen in der Weise kund, dass sie sich vor der Tür aufstellen und ihren Kopf einer ihnen vertrauten Person zuwenden, wobei sie meistens noch einen eigenartigen Laut hören lassen. Diese Kontaktform führt das Tier aus eigenem Antrieb, auf Grund von eigenen Erfahrungen aus. Wie vorsichtig man bei der Deutung dieser Kontaktäusserungen auch vorgehen mag, man wird zugeben müssen, dass es sich um eine Art von spontaner *Kundgebung* handelt, durch welche das Tier versucht, die Aufmerksamkeit auf sich zu lenken und das erstrebte Ziel auf irgendeine uns verständliche Art anzudeuten. Dieser spezifischen Verhaltensweise liegt meiner Ansicht nach eine besondere Funktion zugrunde, die ich „*Aufforderungs-funktion*“ nannte. Unter dieser Funktion wollen wir die Fähigkeit verstehen, an bestimmte Personen durch Andeutung des erstrebten Zieles Wunschäusserungen zu richten und die Personen zu einer dem Wunsch entsprechenden Handlung zu veranlassen.

Ganz analogen Fällen begegnen wir bei Kindern in ihrer vorsprachlichen Periode, wenn

³⁶⁾ G. RÉVÉSZ, Die menschlichen Kommunikationsformen und die sog. Tiersprache. Proc. Ned. Akad. v. Wetensch. A'dam. Vol. 43, 1941.

sie z.B. die Arme nach der Mutter ausstrecken, um auf den Schoss genommen zu werden, oder durch Schreien kundgeben, dass sie aufgehoben werden wollen. Diese und ähnliche Äusserungen setzen die Sprachfunktion noch *nicht* voraus. Das muss betont werden, um jedes Missverständnis über die eigentliche Natur der Aufforderungsfunktion von vornherein auszuschliessen. Die Aufforderungsakte der Kinder und der Tiere stellen keine Sprachakte dar, nicht einmal ihre primitivste Form. Das Zuwenden und Zulaufen, der auffordernde Schrei und Ruf bilden keine Ausdrucksformen der Bezeichnungs-, Darstellungs- oder symbolischen Funktion der Sprache; auch fehlt ihnen jede Ähnlichkeit mit der Struktur der Sprache. Demgegenüber lässt sich nicht leugnen, dass das Tier und Kind auf etwas bestimmtes „hinweisen“, „hindeuten“, etwas „anzeigen“, ihr Verlangen durch sinnvolles Verhalten kund tun und sich dabei an bestimmte Personen wenden mit der nicht zu verkennenden Tendenz, sie bei Erfüllung ihres Verlangens zur Mitwirkung zu veranlassen. Eine gewisse *Mitteilungstendenz* wird man selbst bei grösster Vorsicht der Aufforderungsfunktion nicht absprechen können. Das ist zwar noch keine Sprache, aber jedenfalls eine charakteristische Art der adressierten Kundgebung, der im System der Kontaktformen eine besondere Stellung zuzuerkennen ist, zumal man dieser ursprünglichen Aufforderungsfunktion auch in der ersten Zeit der geistigen Entwicklung des Kindes begegnet, was auf einen entwicklungsgeschichtlichen Zusammenhang der menschlichen Kontaktformen mit den tierischen hinweist..

Unter Umständen richtet sich auch der sprechende Mensch in ähnlicher Weise an bestimmte Personen. Es handelt sich auch hier um eine wortlose, aber bereits um eine *sprachbezogene* Aufforderung, die sich von den *sprachlosen* Aufforderungshandlungen der domestizierten Tiere und Säuglinge wesentlich unterscheidet. Die Differenz liegt nicht nur darin, dass die sprachlose Aufforderungsfunktion sich auf einige elementare vitale Bedürfnisse beschränkt, während die sprachbezogene darüber weit hinausgeht, sondern dass hier eine *Intention auf gegenseitige Verständigung*, ein sog. Verständigungswille, zum Ausdruck kommt, also eine geistige Einstellung, die beim Tier ganz fehlt und beim kleinen Kind noch nicht vorhanden ist. Das domestizierte Tier will nur verstanden werden; es fühlt aber nicht das Bedürfnis, andere zu verstehen. Das sprachbegabte Wesen will indessen nicht nur kundgeben, mitteilen, sondern auch die Kundgabe, die Mitteilung anderer entgegennehmen. Diese Tendenz liegt bei der sprachlosen Aufforderungsfunktion nicht vor. Damit hängt es zusammen, dass das Tier seine eigene Verständigungsform, falls sie von einem anderen Individuum ausgeführt wird, nicht begreift.

Was die Sprache vor allen anderen Kommunikationsformen besonders auszeichnet, ist also die *Intention auf wechselseitigen Kontakt und auf gegenseitige Verständigung*. Diese weit über die vitalen Bedürfnisse hinausgehende *Intention bildet die Grundvoraussetzung der Sprache*. Nicht in den Ausdruckslauten, Ausdrucksbewegungen, auch nicht in der Aufforderungsfunktion als solcher, sondern in der *Intention* zur interindividuellen Verständigung müssen wir den *Urgrund der Sprache* erblicken. Diese Intention findet zwangsläufig die adäquaten Mittel der Verständigung und führt notwendig zur Sprache, und zwar bei solchen Lebewesen, die infolge ihrer psycho-biologischen Gesamtorganisation fähig sind, über das Körperlich-Seelische hinaus in die geistige Sphäre emporzusteigen. Das sind ausschliesslich die Menschen. Den Zeitpunkt, zu welchen jene Intention sich zum erstenmal äusserte, kann man als den Zeitpunkt der Entstehung der Sprache ansetzen. Wie die Intention sich zu allererst kundgab, wissen wir nicht; nach aller Wahrscheinlichkeit in Wortlauten und Gebärdezeichen, die aus differenzierten Naturlauten und Ausdrucksbewegungen entstanden sind. Die Laute und Bewegungen haben demnach im Vorstadium der Sprache eine Rolle gespielt, bloss nicht — wie die früheren Forscher dachten — in spracherzeugendem und sprachbildendem Sinne; sie wurden vielmehr dem Kontaktbestreben dienstbar gemacht, indem sie durch die *spracherzeugende Kraft des Menschen* zu Sprachgebilden transformiert wurden.

Die Intention auf gegenseitige Verständigung ist nicht nur bei der Entstehung der Sprache, sondern auch bei ihrer *Fortbildung* wirksam. Das Bedürfnis, seine Wünsche, Erlebnisse mitzuteilen, drängt den Menschen beständig zur Bildung neuer Worte, neuer

syntaktischer Verhältnisse und grammatikalischer Kategorien. Derselbe Anstoss, welcher die Sprache zur Entstehung brachte und den Urmenschen zum Menschen machte, behält also seine den Fortschritt erzeugende Kraft während der Entwicklung der Sprache bei, folglich während der ganzen Geschichte der Menschheit.

Zusammenfassend und ergänzend können wir sagen: Ausgehend von den einfachsten Kommunikationsformen und geleitet von dem Grundprinzip der Kontakttendenz bzw. des Verständigungswillens kann man eine *Entwicklungsreihe aufstellen, die von der instinktiv-adressierten Äusserungsform über die sprachlose Aufforderung zu der Sprache führt*. Gegen die Möglichkeit dieser Aufeinanderfolge der stets mehr differenzierten und engeren Kontakt herbeiführenden Kommunikationsformen lassen sich meiner Ansicht nach weder entwicklungspsychologisch noch logisch begründete Einwendungen machen. Ob nun die Entwicklung wirklich in der beschriebenen Weise verlaufen ist und ob die angegebenen Kontaktformen wirklich die Entwicklungsstufen in der vorsprachlichen Zeit repräsentieren, ist natürlich nicht zu entscheiden. Die Tatsache, dass beim sprachfähigen Menschen noch immer *alle* Kontaktformen vorhanden sind, könnte nämlich so aufgefasst werden, dass die Sprache nicht unmittelbar den primitiveren Kontaktformen entsprang, wenigstens nicht in dem Sinne, dass die letzteren spurlos in die Sprache übergegangen sind. Es ist denkbar, dass die Sprache gleichsam unabhängig von den primitiveren Ausdrucksformen autochthon zustande kam und sich durch autonome Laut- und Sprachgesetze allmählich entwickelte. Auch bei dieser Deutung der Tatsachen würde die von uns aufgestellte Entwicklungsreihe ihre entwicklungspsychologische Bedeutung behalten; es würde sich dann bei ihr nicht um Stufen der Entfaltung der Sprache, sondern um ihre Vorbedingungen handeln. Die primitiveren Kontaktformen müssten der Sprache jedenfalls vorausgegangen sein, gleichsam ihre biologische Voraussetzung gebildet haben.

Unsere nach dem Prinzip der stetigen Differenzierung aufgestellte Entwicklungsreihe stellt das Gerüst einer *Schichtentheorie der Sprachentstehung* dar. Durch Aufzeigen der Schichten sind natürlich die Uebergänge, vor allem der Uebergang zwischen der sprachlosen Aufforderung und der sprachlichen Verständigungsform, noch nicht verdeutlicht. Wie eine Kommunikationsform sich aus einer anderen entwickelt hat, wie sich der Uebergang von den triebhaften, nichtsbewussten Kommunikationsweisen zu der vergeistigten, sprachlichen, allmählich oder sprungweise vollzogen hat, ist uns nicht bekannt. Wie unsere mutmasslichen halbmenschlichen Vorfahren zu der Sprachtätigkeit vorgegangen sind, wie sie von dem Stadium des Noch-Nicht-Sprechens in das des Sprechens vorgerückt sind, wissen wir nicht. Von der phylogenetischen Entfaltung der Sprache, von dem Geschehen in den ungeheuren Zeiträumen der Entwicklung können wir kein lückenloses Bild entwerfen und das bildet auch nicht die Aufgabe einer entwicklungspsychologischen Theorie der Sprache. Man hat sich damit zu begnügen, auf Grund von tier- und sprachpsychologischen Erfahrungen die wesentlichen Etappen aufzuzeigen, über welche die Entwicklung von den primitivsten Kommunikationsformen, von den nicht-gerichteten über die gerichteten bis zu der höchst entfalteten Kontaktform, zu der Sprache, hat verlaufen können. Diese Stufen des Werdens sind hier, wie ich hoffe, überzeugend dargestellt.

Die von mir aufgestellte Kontakttheorie hat allen anderen Ursprungstheorien gegenüber den Vorteil, dass sie eine von einem einheitlichen und allgemeinen Prinzip, von dem Kontaktsprinzip aus eine entwicklungspsychologisch berechnete Stufenfolge aufzustellen imstande ist. Dies ist den früheren Theorien nicht gelungen. Einen weiteren Vorteil unserer Theorie erblicke ich darin, dass sie auf Vorbereitungsstufen begründet ist, die ohne Ausnahme sowohl beim Tier wie beim Kind und nicht weniger auch beim erwachsenen Menschen vollkommen, ohne die Einzigartigkeit und Autonomie der Sprache im mindesten zu beeinträchtigen. Theoretisch wichtig dünkt mir schliesslich die Feststellung, dass alle von uns unterschiedene Kommunikationsformen beim Menschen sowohl sprachbezogen wie auch in ihrer ursprünglichen, instinktiven Form noch vorhanden sind, so dass die hypostasierten vorsprachlichen Etappen der Sprache noch immer aufzuzeigen sind.

8. *Mensch und Sprache.*

Bei der Aufstellung der Kontakttheorie wurden wir von der Absicht geleitet, bezüglich der Vorgeschichte der Sprache eine entwicklungspsychologisch begründete Lehre zu entwickeln. Ob unsere menschenähnlichen oder halbmenschlichen Vorfahren sich wirklich den dargelegten Kontaktformen bedient haben und ob die Sprache wirklich in der angegebenen Weise entstand, lässt sich, wie gesagt, wegen des Fehlens von empirischen Anhaltspunkten nicht ausmachen. Unsere Theorie erfüllt jedenfalls die Forderungen, welche die Entwicklungslehre an eine brauchbare Theorie stellt: sie rekonstruiert die Entwicklung der Sprachfunktion von den ersten Anfängen bis zu ihrer vollen Ausbildung auf Grund der Tatsachen der vergleichende Psychologie, indem sie die Stufen aufzeigt, welche in immer differenzierter Form zu der letzten Phase der Entwicklung, zu der Sprache, führten. Die Theorie gewinnt dadurch viel an Wahrscheinlichkeit, dass sie nicht nur auf das Prinzip der stetigen Entwicklung, sondern auch auf das des Kontaktes stützt, dem alle Kontaktäusserungen, von den einfachsten bis zu der höchsten, unterworfen sind.

Die Kontakttheorie kann ein Gegengewicht gegen die *Schöpfungshypothese* bilden, welche die Sprache als eine schöpferische Tat des Menschen betrachtet; allein die Kontakttheorie widerspricht nicht, wie es alle übrigen Entwicklungstheorien tun, der Schöpfungshypothese. Ganz im Gegenteil: die beiden Hypothesen ergänzen einander. Man kann sogar einen Schritt weiter gehen und behaupten, dass die Schöpfungshypothese gleichsam einen integrierenden Teil meiner Ursprungstheorie bildet. Die Sache verhält sich nämlich folgendermassen:

Die Vorgeschichte der Sprache hat sich bei den diluvialen Hominiden vollzogen, die stammesgeschichtlich als unsere Ahnen betrachtet zu werden pflegen. Auf diese vorsprachliche Periode bezieht sich die Kontakttheorie als Lehre vom Sprachursprung. Die Sprache als solche jedoch ist, selbst in ihrer primitivsten Form, eine Schöpfung des Menschen. Mensch und Sprache sind unzertrennlich miteinander verbunden. Wir können uns ebensowenig Wandervogel ohne Saisonflug oder Bienen ohne soziale Organisation vorstellen wie Menschen ohne Sprache. Die soziale Organisation, die gemeinsame Nahrungssuche, die Brutpflege, der Bau der Zellen, die gegenseitige Hülfe sind für die Biene biologisch genau so notwendig und wesentlich wie für den Menschen die Sprache als Grundlage seines sozialen Daseins. Biene und Bienenstaat, Storch und Flug, Katze und Jagdtrieb sind genau so unzertrennlich miteinander verbunden wie Mensch und Sprache. Die Sprache gehört zum Wesen des Menschen. Die Frage, ob der Mensch oder die Sprache früher ist, gleicht der alten Philosophenfrage, ob das Ei oder das Huhn früher ist. Ohne eine Art Schöpfungsbegriff kommen wir selbst in der Biologie nicht aus: der Mensch *schuf* die Sprache, und die Sprache *bildete* den Menschen *aus*, machte ihn zum Menschen. Die Sprache ist eine Schöpfung der geistigen Natur des Menschen und lässt sich als eine nach folgerechten, unabänderlichen Gesetzen der menschlichen Natur entstandene Tätigkeit verstehen, mithin als eine Tätigkeit, die sich aus sich selbst entfaltete.

Der Mensch hat immer gesprochen. Als er noch nicht sprach, war er eben noch kein Mensch. Sollte es uns auch einmal gelingen, lückenlos die Ahnenreihe des Menschen anatomisch aufzubauen, die sog. Stammesgeschichte des Menschen mit ihren noch fehlenden Uebergangsformen zu rekonstruieren: das Problem des Ursprungs der Sprache würde dadurch seiner Lösung nicht näher gebracht werden. Erblickt man in *Pithecanthropus erectus* oder in irgendeiner früheren Spezies der menschenähnlichen Affen den Vorfahren des Menschen, so bleibt die Frage noch immer offen, ob jener Uebergangstyp mit Sprachfunktion begabt gewesen ist oder nicht. Könnte diese Frage in positiver Weise beantwortet werden, z.B. durch den einwandfreien Nachweis, dass der *Pithecanthropus* in Java Zeichnungen hinterlassen hätte, die ohne Sprachfunktion nicht hätten zustande kommen können, dann müssen wir sagen, dass der *Pithecanthropus* eben ein Mensch war. Fällt die Antwort negativ aus, dann war er ein Affe und kein Mensch.

Wie man auch immer das Problem stellt, dehnt, deutet, wir können nicht umhin, uns den Menschen von Anfang an als eine geistige und sprachbegabte Persönlichkeit vorzu-

stellen. Mag dieser Mensch seine Gedanken und Wünsche auch noch so primitiv zum Ausdruck gebracht haben: seine Mitteilungsform kann nur *die Sprache* gewesen sein. Es gibt gewiss Bedingungen, die erfüllt sein müssen, damit der Mensch sich einer Sprache — aktiv oder passiv — zu bedienen vermag; die Einsicht in diese Bedingungen und in die Notwendigkeit ihrer Erfüllung gelingt indessen nur durch Anknüpfen an die schon *fertige* und *gesprochene* Sprache. Aus der Natur und Struktur der lebendigen Sprache können wir jene Bedingungen durch Analyse ableiten; aber wir können die Sprache nicht aus ihnen *aufbauen*. Die Sprache und die sie bestimmenden Funktionen bilden eine *Einheit*. Anderwärts will ich den Beweis liefern, dass alle diese Funktionen erst *durch* die Sprache ihren Charakter und ihre spezifischen Eigentümlichkeiten erhalten und dass *ohne* die Sprache die meisten dieser Funktionen nicht bestehen, geschweige denn sich zu entfalten vermögen. Aus den der Sprache zu Grunde liegenden Funktionen kann man also meiner Ansicht nach die Sprache nicht ableiten, wohl aber umgekehrt *aus der Sprache alle fundamentalen spezifisch menschlichen Funktionen* ³⁷⁾.

Aus diesen Ueberlegungen geht die Berechtigung der Schöpfungshypothese im Rahmen der Entwicklungsgeschichte klar hervor. Gehen wir von der logisch wie entwicklungsgeschichtlich begründeten Annahme der Einheit des *sprechenden Menschen* aus, einer Annahme, die kaum einen Widerspruch erwecken kann, dann gelangen wir zu dem Ergebnis, dass die Kontakttheorie mit der Schöpfungstheorie gut vereinbar ist. Beide Theorien haben verschiedene Aufgaben zu lösen: die Kontakttheorie versucht, die *Vorgeschichte* der Sprache, die sich vor der Menschwerdung vollzogen hat, zu rekonstruieren, die Schöpfungshypothese beabsichtigt die *Entstehung* und *Entfaltung* der Sprache aus der geistigen Natur des Menschen abzuleiten. Zwischen beiden Theorien stellt das Prinzip der gegenseitigen Verständigung die Verbindung her, welches die Triebfeder des menschlichen Schöpfungsaktes anzeigt.

Die Kontakttheorie stellt in ihrer Verbindung mit der Schöpfungshypothese einen Fortschritt in der Sprachursprungsforschung dar. Dass die Lücke zwischen dem sprachlosen und dem spracherfüllten Stadium durch die Theorie nicht überbrückt wird, liegt nicht an ihr, sondern an dem Umstand, dass ein stetiger Uebergang zu der Sprache ebensowenig aufzeigbar ist wie ein Uebergang zwischen Bewussten und Nichtbewussten, zwischen Trieb und durch Zielvorstellungen gerichteten Willen. Es liegt eine Illusion vor, wenn man glaubt, dass eine tierische Reaktionsweise mit einer äusserlich zwar ähnlichen, dem Wesen nach aber ganz verschiedenen menschlichen Verhaltensweise entwicklungsgeschichtlich zu verkoppeln und dass die letztere ohne Weiteres als eine natürliche Um- oder Weiterbildung der ersteren zu betrachten sei. Bei einer solchen Auffassung lässt man völlig ausser Acht, dass alle diese Funktionen an die *Geistigkeit* des Menschen gebunden sind; diese entwicklungsgeschichtlich zu verfolgen ist aber nicht möglich. Von hier aus betrachtet, bleibt nichts anderes übrig, als das Problem der Entstehung der Sprache geradezu als ein unlösbares Problem anzusehen. Die Behauptung der Unlösbarkeit bedeutet in diesem Zusammenhang nicht die Behauptung der Unmöglichkeit einer stetigen Entstehung, sondern nur die Behauptung der Notwendigkeit eines Verzichtes auf Ableitung eines höheren und qualitativ anderen aus einem Primitiveren und Verschiedenartigen.

Wir dürfen unsere Ziele nicht zu hoch stellen; wir müssen uns daran genug sein lassen, die entwicklungspsychologische Betrachtungsweise auch auf das Ursprungsproblem der Sprache angewendet, die Formen des gegenseitigen Kontaktes als Etappen einer Entwicklung erkannt, und schliesslich alle diese Formen, von den einfachsten bis zu der am reichsten ausgestalteten, als durch ein einziges Prinzip beherrscht verstanden zu haben. Dass wir nicht imstande sind uns von den Uebergangsstadien, die von den adressierten sprachlosen Kontaktäusserungen zur Sprache führen, eine plausible Vorstellung zu bilden, liegt darin, dass *vom Tier kein gerader Weg zum Menschen führt*.

³⁷⁾ Die einzige Ausnahme bildet das Denken, das aber mit der Sprache eine unzertrennliche Einheit bildet, folglich voneinander nicht ableitbar sind.

Comparative Physiology. — *Die angebliche Diffusion von Glykocholat-Oelsäurelösungen durch Pergament.* Von H. J. VONK und C. J. A. M. ENGEL. (Aus dem Laboratorium für vergleichende Physiologie der Universität Utrecht.) (Communicated by Prof. H. J. JORDAN.)

(Communicated at the meeting of March 28, 1942.)

Von den bei der Fettspaltung im Darme entstehenden Produkten sind die Fettsäuren in Wasser und verdünnten Salzlösungen unlöslich. Man stellte sich seit PFLÜGER's Untersuchungen¹⁾ vor, dass diese Säuren als Seifen durch die Darmwand gehen. In PFLÜGER's Zeit war über die Wasserstoffionenkonzentration des Darminhaltes und über die physikalisch-chemischen Existenzbedingungen von Seifenlösungen noch fast nichts bekannt. Als es sich nun später herausstellte, dass bei dem pH des Darminhaltes Seifenlösungen nicht bestehen können, war die Möglichkeit eines Transportes von Fettsäuren als Seifen hinfällig geworden. Nun hatte aber PFLÜGER, ehe er zu der Vorstellung kam, dass die Fettsäuren hauptsächlich als Seifen die Darmwand durchqueren sollten, schon beobachtet, dass Galle im Stande ist ausser Seifen auch Fettsäuren in Lösung zu bringen. Auch andere Untersucher stellten diese Tatsache fest. VERZÁR nam 1929 diese Frage wieder auf²⁾ und untersuchte, bei welchen pH-Werten diese lösende Wirkung von Galle und gallensauren Salzen auf Fettsäuren möglich ist. Diese pH-Grenzen waren derart, dass man eine lösende Wirkung der Galle auf Fettsäuren im Darminhalte annehmen muss. VERZÁR hat darauf die Frage zu beantworten versucht, ob bei der Dispersion der höheren Fettsäuren in Galle und in Lösungen von gallensauren Salzen eine molekulare oder eine kolloidale Lösung entsteht. Er hat dazu Dialysier- und Filtrationsversuche angestellt und fand, dass die Fettsäuren tatsächlich durch Diffusionshülsen No. 579 von SCHLEICHER und SCHÜLL hindurch diffundierten. Das Ergebnis dieser Versuche schien uns etwas unwahrscheinlich zu sein, einmal aus theoretischen Gründen und dann weil einer von uns³⁾ beim Arbeiten mit hydrotropem⁴⁾ Verdauungssaft von Invertebraten (Krebsen) eine derartige Diffusion von Oelsäure nicht feststellen konnte.

Es schien uns ziemlich unwahrscheinlich, dass in Vitro in so einfacher Weise, ohne jede Inanspruchnahme irgendwelcher chemischer Manipulationen in wässriger Lösung eine chemische Verbindung zwischen Fettsäure und gallensaurem Salz zustande kommen sollte. VERZÁR hat sich auch nicht über die mutmassliche molekulare Struktur dieser „Verbindung“ ausgesprochen. Er erwähnt nur, dass der „Komplex“ bei Kataphoreseversuchen negative Ladung zeigt. Bei einem Ueberschuss von Fettsäure sollen grössere Komplexe entstehen, welche eine grössere Anzahl Fettsäuremoleküle auf je ein Molekül Natriumtaurocholat oder -Glykocholat enthalten.

Alle diese Tatsachen scheinen uns mehr in der Richtung einer kolloidchemischen Deutung der Erscheinung zu weisen. Gallensaure Salze sind stark oberflächenaktiv und adsorbierbar. Dispergiert man heisse oder in Alkohol gelöste Fettsäuren in einer Lösung von Gallensäuren, so können die Moleküle der letzteren an den Fettsäureteilchen adsorbiert werden und ihnen eine negative Ladung erteilen, welche der Stabilitätsfaktor der auf diese Weise entstandenen kolloidalen Teilchen ist. Es ist klar, dass eine solche Lösung nicht durch Pergamenthülsen No. 579 SCHLEICHER und SCHÜLL diffundieren würde.

1) E. PFLÜGER, Pflügers Archiv 80, 81, 82, 85, 86, 88, 89, 90.

2) F. VERZÁR und A. KÚTHY, Bioch. Zs. 205, 369 (1929), 210, 265 u. 281 (1929).

3) H. J. VONK, Zeitschr. f. Physiol. 21, 717 (1935).

4) NEUBERG (Bioch. Zs. 76 (1916) hat die Erscheinung, dass ein bestimmter Stoff einen anderen, an sich unlöslichen Stoff, in Wasser löslich macht, Hydrotropie genannt.

Um diese Frage zu lösen, stellten wir Versuche über die Diffusion derart „gelöster“ Oelsäure an, welche der Gegenstand einer früheren Mitteilung waren¹⁾. Es stellte sich heraus, dass eine derartige Diffusion nicht stattfindet und dass die von VERZÁR und Mitarbeitern erhaltenen Resultate auf unzulänglicher Versuchstechnik beruhen. Wir fanden, dass die richtige Bestimmung von Fettsäuren in Gegenwart von einem Ueberschuss an gepaarten Gallensäuren keineswegs einfach ist. Man muss den pH der Lösung bei der Aetherextraktion genau einstellen und die Bestimmung ist nur ausführbar, wenn man daneben eine Blankoextraktion mit Gallensäure anstellen kann.

Unabhängig von uns kam BREUSCH²⁾ ungefähr gleichzeitig zum gleichen Resultat. Er benutzte eine ganz andere Fettsäurebestimmung, wobei diese Säuren sublimiert wurden.

Die Sache hätte als erledigt gelten können, wenn nicht QUAGLIARIELLO und CEDRANGOLO³⁾ 1938 Versuche veröffentlicht hätten, aus welchen hervorzugehen scheint, dass doch eine Diffusion von Fettsäuren und selbst von Triolein stattfinden kann, wenn diese Stoffe in Gegenwart von Galle oder Gallensäuren gelöst werden. Nun sind in diesen Versuchen grösstenteils Zellophanmembranen benutzt worden, welche eine ganz andere Permeabilität aufgewiesen haben können, als die von VERZÁR und uns benutzten Membranen No. 579 SCHLEICHER & SCHÜLL. Doch auch bei den letztgenannten Membranen (die Nummer wird nicht angegeben) wurde von QUAGLIARIELLO Durchgang von Fettsäuren gefunden. Die Fettsäuremengen in der Innen- und Aussenflüssigkeit wurden elektrometrisch titriert bis zu pH 8,5. Aus dem Unterschied mit der Kontrolle wurde dann die Fettsäuremenge in Innen- und Aussenflüssigkeit ermittelt. Der totale Lipoidgehalt wurde mit der Methode von KUMAGAWA-SUTO bestimmt. Leider wurde die genaue Zusammensetzung der Flüssigkeiten nicht angegeben. Es wird weder der pH der Flüssigkeit, noch der Anfangsgehalt der Innenflüssigkeit an Fettsäure oder Glyzerid, noch die Konzentration der benutzten Gallensäure erwähnt. Das Fehlen aller dieser Zahlen macht natürlich eine Beurteilung von der Zuverlässigkeit dieser Versuche äusserst schwierig und macht es auch unmöglich sie unter genau gleichen Versuchsbedingungen zu wiederholen. Wohl ist aber den Zahlen zu entnehmen, dass ziemlich grosse Mengen Fettsäure zugesetzt wurden. Wenn 40–60 mg Oelsäure von 10 cm³ Dispersion (innen) in 40 cm³ Dispersionsmittel (ausser) übergang, so muss wenigstens das fünffache dieser Mengen (also 200–300 mg) in die 10 cm³ Innenflüssigkeit dispergiert worden sein. Weiter ist es fraglich, ob die beiden von Q. verwendeten Methoden zur Fettsäurebestimmung genügende Genauigkeit haben, besonders wenn so starke Emulgatoren wie die Gallensäuren, störend wirken können. Welchen Schwierigkeiten man hier begegnen kann, geht aus unserer vorigen Abhandlung hervor. In QUAGLIARIELLO's Abhandlung wird auch nicht erwähnt, ob die benutzte Methode besonders für diesen Zweck ausprobiert worden war.

Nach dem Erscheinen von QUAGLIARIELLO's kleiner Arbeit, schien uns eine Neubearbeitung der Frage wiederum erwünscht und wir haben diese teilweise mit der in unserer vorigen Abhandlung ausgearbeiteten Methode vorgenommen. Zuerst haben wir uns nach einer etwas einfacheren Methode der Fettsäurebestimmung umgesehen, welche weniger zeitraubend ist als die erstgenannte. Dazu benutzten wir die Tatsache, dass bei Dispersion von Fettsäuren in Lösungen von gallensauren Salzen die Oberflächenspannung erniedrigt wird. Zwar ist diese Erniedrigung lange nicht so gross als die, welche durch die Lösung von gallensaurem Salz in Wasser stattfindet, aber sie genügt doch um durch stalagmometrische Messung einen (sei es auch nicht sehr feinen) Massstab für die Fettsäuremenge zu ergeben. Bei dieser Methode fallen also alle umständliche Extraktionen, Trocknungen und Wägungen unserer früheren Methode fort.

Zuerst stellten wir einige Versuche über die Verwendbarkeit der Bestimmung der Tropf-

1) H. J. VONK, CHR. ENGEL und C. ENGEL, *Biochem. Zs.* **295**, 171 (1938).

2) F. L. BREUSCH, *Bioch. Zs.* **293**, 280 (1937).

3) G. QUAGLIARIELLO e F. CEDRANGOLO, *Rendiconti, d. R. Accad. nazion. dei Lincei* **27**, 503 (1938).

zahl für die Ermittlung der Oelsäuremenge an. Verwendet wurde 20 cm³ einer 1.5 % Lösung von Natriumglykocholat in $\frac{M}{15}$ Phosphatpuffer von pH 6.81. Verschiedene Mengen einer 5 %-tigen alkoholischen Oelsäurelösung wurden hieran zugesetzt. Es wurde kontrolliert, ob die kleine Menge des hieran zugesetzten 96 %-tigen Alkohols an sich eine Änderung der Tropfzahl herbeiführte. Dies war nicht der Fall:

a. Wasser	Tropfzahl: 42.7	} Volumen Glykocholat- lösung 50 cm ³ .
b. Glykocholat	„ : 61.4	
c. b mit 0.62 cm ³ 96 % Alkohol	„ : 61.4	
d. b mit 0.62 cm ³ 5 % Oelsäure in 96 % Alkohol (31.0 mg Oelsäure)	„ : 67.7	

Die Lösung c wurde mit 2 cm³ 1 N Salzsäure angesäuert und darauf 2 mal mit (50 bzw. 40 cm³) Aether ausgeschüttelt. Nach Neutralisieren mit 2 cm³ 1 N Natronlauge und Entfernung der gelösten Aethermenge durch 3-stündiges Aufblasen eines Luftstromes, wurde die Tropfzahl abermals bestimmt und betrug nun 62.1. Die Lösung d, ebenso behandelt, ergab als Tropfzahl 61.9. Hieraus geht hervor, dass durch diese Behandlung die Oelsäure praktisch ganz wieder aus dem Gemisch entfernt werden kann. Schon diese Tatsache spricht gegen das Entstehen einer festen Bindung des Glykocholats an die Oelsäure und für eine oberflächliche Anlagerung.

Weiter kontrollierten wir, ob die Zunahme der Tropfzahl der Menge der zugesetzten Fettsäure proportional war. Hierzu wurden Mengen Oelsäure von 2.5 bis 25 mg an 20 cm³ der oben erwähnten Natriumglykocholatlösung zugesetzt. Das Ergebnis dieses Versuches wurde in Fig. 1 abgebildet. Von 0 bis 12.5 mg besteht tatsächlich diese Proportionalität.

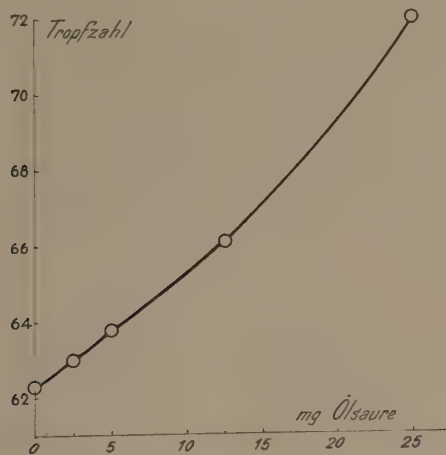


Fig. 1. Erniedrigung der Oberflächenspannung einer 1.5 %-tigen Lösung von Natriumglykocholat bei Zusatz von verschiedenen Mengen Oelsäure. Bestimmung der Oberflächenspannung stalagmometrisch.

Bei grösseren Mengen (25 mg) zeigt sich eine kleine Abweichung (dadurch, dass dort die Alkoholmenge eine Rolle mitspielt).

Die Methode ist also für unsere Zwecke brauchbar. Die Abweichung bei etwas grösserer Menge wurde dadurch ausgeglichen, dass der Kontrolllösung die gleiche Alkoholmenge ohne Fettsäure zugesetzt wurde.

Wir müssen allerdings bemerken, dass die Erniedrigung der Oberflächenspannung durch den Zusatz einer gleichen Menge Oelsäure verschieden ausfallen kann, je nach der Art

der Dispersion (heiss oder kalt; Zusatz der Gallensäure vor oder nach der Dispersion der Oelsäure).

Diffusionsversuche. Eine Reihe dieser Versuche stellten wir an, bei welchen die Messung der Tropfzahl als Kriterium für das Stattfinden von Diffusion benutzt wurde. Kleine Diffusionshülsen von 10 cm³ Inhalt wurden hierzu verwendet. Die fettsäurehaltige Lösung (20 cm³) war Aussenflüssigkeit, die fettsäurefreie (10 cm³) Innenflüssigkeit. Das Dispersionsmittel bestand aus einer Lösung von 1.5 % Na-Glykocholat in $\frac{M}{15}$ Phosphatpuffer. Das Niveau der Innen- und Aussenflüssigkeit befand sich in gleicher Höhe. Die der Aussenflüssigkeit zugesetzte Fettsäuremenge betrug 12.5 mg. Die Versuche dauerten 48 Stunden.

Tabelle I gibt eine Uebersicht der erhaltenen Resultate. Beim Stehen während 48 Stunden änderte sich bisweilen die Oberflächenspannung der Kontrolllösung einigermaßen. Darum wurde sowohl die Tropfzahl vor und nach Anfang des Experimentes in einem Blanko-versuch bestimmt.

Aus der Tabelle I ist ersichtlich, dass der Unterschied zwischen den Tropfzahlen von

TABELLE I.

	Tropfzahlen			PH
	Aussenflüssigkeit (I)	Innenflüssigkeit (II)	Differenz von I und II	
Kontrolle zu Anfang.	67.0	62.3	4.7	6.81
Kontrolle nach 48 St.	66.1	62.4	3.7	
Diffusionsversuch I .	66.3	62.7	3.6	
Diffusionsversuch II .	66.4	62.7	3.7	
Kontrolle zu Anfang.	66.2	63.1	3.1	6.98
Kontrolle nach 48 St.	66.0	63.2	2.8	
Diffusionsversuch I .	66.2	63.4	2.8	
Diffusionsversuch II .	66.3	63.4	2.9	
Kontrolle zu Anfang.	63.8	62.0	1.8	6.98
Kontrolle nach 48 St.	63.9	62.4	1.5	
Diffusionsversuch I .	64.0	62.4	1.6	
Diffusionsversuch II .	64.0	62.4	1.6	
Kontrolle zu Anfang.	64.0	62.2	1.8	6.78
Kontrolle nach 48 St.	64.2	62.4	1.8	
Diffusionsversuch I .	64.2	62.4	1.8	
Diffusionsversuch II .	64.3	62.4	1.9	
Kontrolle zu Anfang.	64.2	62.3	1.9	6.66
Kontrolle nach 48 St.	63.3	62.4	1.9	
Diffusionsversuch I .	64.3	62.4	1.9	
Diffusionsversuch II .	64.4	62.4	2.0	
Kontrolle zu Anfang.	66.1	62.3	3.8	6.80
Kontrolle nach 64 St.	65.3	62.4	2.9	
Diffusionsversuch . .	65.4	62.4	3.0	

Innen- und Aussenlösung während der Dauer des Versuches bestehen bleibt und dass also keine nennenswerte Diffusion stattfindet.

Weiter stellten wir noch einige Versuche an, wobei die Fettsäuremenge mittels einer

der in der genannten vorigen Abhandlung von uns ausgearbeiteten Methode bestimmt wurde. Hierbei wurde also die zu untersuchende Lösung bis zu einem bestimmten pH angesäuert, mit Petrolaether ¹⁾ ausgeschüttelt, der Petrolaetherextrakt getrocknet, verdampft und gewogen. Für die genaue Ausführung verweisen wir auf die genannte Abhandlung. Die gewogene Fettsäuremenge wurde weiter in Alkohol gelöst und unter Zusatz von Phenolphthalein titriert. Dieses Verfahren hatten wir früher noch nicht angewandt; in Kontrollversuchen stellte sich heraus, dass es gute Resultate lieferte. Die Resultate geben wir in Tabelle II und III. In den darin angeführten Zahlen sind Korrekturen verarbeitet für die Blankowerte der Reagenzien und für die Tatsache, dass man die eingebrachten Flüssigkeiten nicht völlig wieder aus den Räumen in und um die Hülse zurückerhalten kann.

TABELLE II.

Lösung Na-Glykocholat 1.5 % in $\frac{M}{15}$ Phosphatpuffer. p_H 6.81. Innen 50 cm³ dieser Lösung; aussen 50 cm³ dieser Lösung mit 31.25 mg Oelsäure. Zur Analyse nach 48 St. Diffusion 20 cm³ mit 12.5 mg Oelsäure (Doppelbestimmungen). Alle Zahlen bedeuten mg.

	Kontrolle	Diffusionsversuch I		Diffusionsversuch II	
		Aussen	Innen	Aussen	Innen
Nach Gewicht	10.95 } Mittel 11.35 } 11.15	9.85 } 10.95 } 10.40	— 0.05 } 0.65 } 0.3	10.45 } 11.65 } 10.55	0.15 } 0.55 } 0.35
Nach Titration	12.7 } 12.6 } 12.65	11.0 } 11.3 } 11.15	0.1 } 0.2 } 0.15	10.6 } 10.5 } 10.55	0.3 } 0.1 } 0.2

TABELLE III.

Lösung Na-Glykocholat 1.5 % in $\frac{M}{15}$ Phosphatpuffer. p_H 6.81. Innen 10 cm³ dieser Lösung mit 50 mg Oelsäure; aussen 20 cm³ dieser Lösung ohne Oelsäure. Analysiert wurden (nach 48 St.) die totalen Mengen aussen und innen. Alle Zahlen bedeuten mg.

	Diffusionsversuch I			Diffusionsversuch II		
	Innen	Aussen	Total zurück-gefunden	Innen	Aussen	Total zurück-gefunden
Nach Gewicht	36.2	5.0	41.2	39.8	4.7	44.5
Nach Titration	36.2	4.7	40.9	41.5	4.8	46.3

Die Versuche von Tabelle II schlossen sich den Versuchen unserer vorigen Abhandlung an, welche mit relativ kleinen Mengen Fettsäure auf relativ grosse Mengen Glykocholat ausgeführt wurden. Hierbei kam die Fettsäure in der Aussenflüssigkeit, was VERZAR's Versuchsanordnung entsprach. Es wurden je 20 cm³ der Innen- und Aussenflüssigkeit analysiert; die Flüssigkeit mit Oelsäure enthielt davon 12.5 mg auf 20 cm³. Auf diese Weise konnten Doppelbestimmungen ausgeführt werden.

Die Versuche von Tabelle III wurden im Anschluss an die Versuche QUAGLIARIELLOS so ausgeführt, dass die Oelsäure (50 mg) in der Innenflüssigkeit (10 cm³) dispergiert

¹⁾ Früher wurde mit Äther ausgeschüttelt.

wurde. Die angewandte Menge Oelsäure war also hier, ebenfalls im Anschluss an QUAGLIARIELLOs Anordnung, ziemlich gross. Jedoch nicht so gross als bei diesem Forscher, da bei den grossen von ihm verwendeten dispergierten Mengen keine klare (wenn auch vielleicht kolloidale) Lösungen mehr entstehen.

Aus der Tabelle II ist erstens zu entnehmen (Kontrolle), dass von den zugesetzten Fettsäuremengen nur 80 bis 90 % zurückgefunden wurden, sowohl mit der Gewichts- als mit der Titrationsmethode. Diese Resultate sind etwas ungünstiger als die unserer vorigen Abhandlung. In Anbetracht der sehr grossen Differenzen, welche zwischen Innen- und Aussenflüssigkeit nach 48-stündiger Versuchsdauer gefunden werden, ist diese Genauigkeit aber durchaus genügend um einen Schluss auf das Stattfinden einer eventuellen Diffusion ziehen zu können. Dieser Schluss muss dann für die Tabelle II lauten, dass bei diesen zugesetzten Mengen Oelsäure keine Diffusion stattfindet. Die sehr kleinen Zahlen, welche für die Innenflüssigkeit gefunden werden, sind als blosse Versuchsfehler zu betrachten.

Nicht ganz zu dem gleichen Schluss können wir für die Resultate der Tabelle III kommen. Aus dieser Tabelle würde man den Schluss ziehen können, dass in dieser Zeit (48 St.) und bei dieser Versuchsanordnung (relativ viel Oelsäure) tatsächlich etwa 10 % der zurückgefundenen Menge Oelsäure durch die Membran passiert ist. Es ist nun die Frage, ob dieser Schluss gerechtfertigt ist und also doch mit einer geringen Diffusion gerechnet werden muss. Tatsache ist, dass wir von dem genauen kolloidalen Zustand der „Lösung“ von Fettsäure in Lösungen von gallensauren Salzen, sehr wenig wissen¹⁾. VERZÁR hat behauptet, dass bei kleinen Oelsäuremengen molekulare Dispersion besteht, bei grösseren Mengen die Teilchengrösse aber zunimmt. Nach unseren früheren Versuchen und nach den Untersuchungen von BREUSCH, können wir aber annehmen, dass dies nicht der Fall ist. Denn gerade wenn kleine Mengen Fettsäure dispergiert wurden fanden wir keine Diffusion. Wenn die Versuche QUAGLIARIELLO's und unsere Versuche von Tabelle III richtig sind, so könnten wir höchstens annehmen, dass eine kleine Fraktion der Oelsäure-Gallensäure Komplexe molekulardispers ist. Verwendet man dann eine kleine Menge Oelsäure, so würde diese diffusible Fraktion der Analyse entgehen. Nimmt man aber grössere Oelsäuremengen, so könnte diese kleine Fraktion der Analyse zugänglich werden. Wie schon bemerkt, hat QUAGLIARIELLO die Fettsäuremengen, welche er dispergierte, nicht angegeben. Wir wissen also nicht, ob seine Zahlen für die durchgewanderten Mengen bedeuten, dass nur eine Fraktion der Oelsäure durchgewandert ist, oder dass die Konzentrationen innen und aussen gleich geworden sind, wie VERZÁR für seine Versuche angab. Der von uns erhaltene Diffusionseffekt ist — falls er sich in weiteren Versuchen bestätigen lässt — sehr gering, zumal da die Versuchsdauer 4 Mal länger ist als derjenige in VERZÁR's Versuchen. In dieser Zeit passiert in unseren Versuchen nur ein Zehntel der Oelsäuremenge, während in VERZÁR's Versuchen in 12 Stunden die Konzentration innen und aussen gleich geworden sein sollte. Nach unserer letzteren Abhandlung und nach BREUSCH's Versuche ist aber VERZÁR's Resultat einer unzulänglichen Fettbestimmung zuzuschreiben.

Mit Rücksicht auf die hohe Diffusionswerte, welche QUAGLIARIELLO findet, ist noch folgendes zu erwägen. Es ist schwer Oelsäure chemisch rein zu erhalten. Gesättigte und ungesättigte Fettsäuren können besonders als Verunreinigungen auftreten. Nun passieren nach BREUSCH niedere Fettsäure und stark ungesättigte Fettsäure, wenn sie mit Natriumglykocholat gelöst sind, Pergamentmembranen. Stark unreine Oelsäurepräparate können also tatsächlich einen Diffusionseffekt vortäuschen. In unseren Versuchen von Tabelle III kann diese Möglichkeit aber nicht in Frage kommen, da wir mit einem von uns nach der Methode von BERTRAM²⁾ besonders gereinigtem Oelsäurepräparat arbeiteten.

¹⁾ Ueber niedrige Fettsäuren liegt eine Arbeit von HOLWERDA (Bioch. Zs. 294, 372 (1937)) vor. Vgl. auch Holwerda Bioch. Zs. 296, 1 (1938).

²⁾ Rec. Trav. chim. Pays-Bas 46, 397 (1927).

DISKUSSION.

BREUSCH ¹⁾ meint, „dass die Gallensäuren für die Resorption der Lipoide im Darm bei weitem nicht die Rolle spielen, die man ihnen bis jetzt zugedacht hat“, und dass sie nur für die Wirkung der Lipase von Bedeutung sind. Mit dieser Schlussfolgerung können wir aber nicht übereinstimmen. Auch wenn die Fettsäuren nur kolloidal gelöst sind, muss die Bedeutung der Gallensäuren zur Vorbereitung der Resorption gross sein. Makroskopische Partikel werden niemals ²⁾ vom Darmepithel der Vertebraten aufgenommen. Jedes Agens das die Teilchengrösse eines für Resorption bestimmten Stoffes verringern kann, muss also diese Resorption begünstigen.

BREUSCH ³⁾ behauptet „dass weiterhin manche Tiere, z.B. Frösche kein Gallensystem haben und trotzdem Fett resorbieren“. Diese Behauptung ist irrtümlich: die Galle fehlt bei keiner einzigen Vertebratengruppe ⁴⁾ mit Ausnahme der erwachsenen Cyclostomen.

IVY ⁵⁾ hat bemerkt, dass der Befund, ob Fettsäure (in Lösung gebracht mit Gallensäure) durch bestimmte Membranen diffundiert oder nicht, nichts aussagt über ihre Diffusibilität durch die Darmwand. Dieser Bemerkung können wir völlig zustimmen. Deshalb ist der Befund aber nicht unwichtig. Er sagt etwas aus über die Teilchengrösse der Komplexe. Seit der Entdeckung des Erepsins und seiner Teilenzyme (die Disaccharasen waren schon früher bekannt) schien es alsob alle Nahrungsstoffe im Darne vor ihre Resorption zu sehr kleinen Molekülen abgebaut werden müssen. Mit Hinsicht darauf ist es nun interessant zu wissen, dass Stoffe mit weit grösseren Teilchen recht wohl zur Resorption gelangen können. Die Spaltung von Eiweiss und Kohlenhydraten muss also noch eine andere Bedeutung haben als die starke Verringerung der Molekülgrösse. Für die Zuckerresorption weisen die neueren Ergebnisse (dass die Möglichkeit der Phosphorylierung über die Aufnahme entscheidet) auch wohl in diese Richtung. Ausserdem sind höhere Eiweissprodukte bekanntlich giftig für den Organismus. Auch mit Hinsicht darauf ist die Zerkleinerung also von Bedeutung.

1) l.c. und zwar S. 292 unten.

2) Mit Ausnahme einer Phagozytose von chinesische Tusche, welche von VON MÖLLEN-DORF bei jungen Mäusen beobachtet wurde.

3) l.c. und zwar S. 293.

4) Die Gallenblase kann fehlen, ist aber gerade bei Fröschen tatsächlich vorhanden.

5) Annual Rev. of Physiology 1, 253 (1939).

Medicine. — *The audiogram in diseases of the transmission apparatus of the ear, especially in cases of external otitis*¹⁾. By G. DE WIT. (Communicated by Prof. A. P. H. A. DE KLEYN.)

(Communicated at the meeting of March 28, 1942.)

The causes of deafness can be divided into two groups. The most essential and to the therapy the least accessible cases of deafness are those, caused by a disturbance in the perceptive apparatus itself or in the higher paths included the hearing centrum, termed: nervous, internal ear- or perception deafness.

The second group comprises all forms in which not the perception but the transmission of sound waves from the outer world to the perceptive organ is disturbed: transmission deafness.

Since here a pure physical occurrence is present, these cases of deafness not only have a more simple and surveyable form but can also more easily be influenced. They gave partly an accessible field for research and are examined more extensively since the audiometer enabled us to obtain in a simple way complete data about the whole tone-region covered by the hearing.

This transmission apparatus consists of the auditory canal and middle ear, as far as the air conduction-, and of perilymph and endolymph as far as the transmission fluid is concerned.

Between external and middle ear lies the membrana tympani, as a link belonging to both, although most intimately united with the middle ear. It is connected with the ossicular chain of which one ossicle, the stapes, closes the foramen ovale. The foramen ovale together with the bony promontorium and the foramen rotundum forms the separation between middle ear and transmission fluids.

The disturbances of hearing of the second group can be localised in each of these parts.

It is certainly worth while to investigate the different localisations of the cause of the disturbed hearing in the mentioned parts. This might also supply material for a refined diagnosis about the localisation of a given deafness.

Diminished auditory acuteness by disturbances in the peri- and endolymph is not known with certainty; at best they are supposed by some investigators; hence, their existence being hypothetical we will pass them without comment.

So much the better the disturbances of the next link — reckoned from the internal ear —: the oval window, are known.

The peculiar changes, known as otosclerosis, cause deafness if the process is localised at the border of the oval window. This deafness is due to a stiffening, developing in the stapes-vestibulum symphysis by which the stapes (last link in the ossicular chain) is impeded in its movements.

Formerly this deafness was described as a typical bass-deafness. Indeed some cases of bass-deafness are found in this disease: a rare form of deafness, only present in this affection (1).

However in most cases "Bass-deafness" demonstrated with tuning forks, appears on audiometric examination, not to be a bass-deafness at all, but a deafness for all tones. Without going into details the following explanation can perhaps be given for this controverse: a low tuning fork has a small (physical) elementary intensity but a long vibration

¹⁾ These investigations have been made possible by a support of the Government of the Netherlands, from the proceeds of the sale of "Zomerpostzegels".

time. A high tuning fork, in the contrary, begins with a great intensity but has a steep decrement. If the threshold of hearing is increased, a larger part of the vibration time of the low tuning fork is cut than of the high one.

Here follows an example of a bass-deafness in otosclerosis.

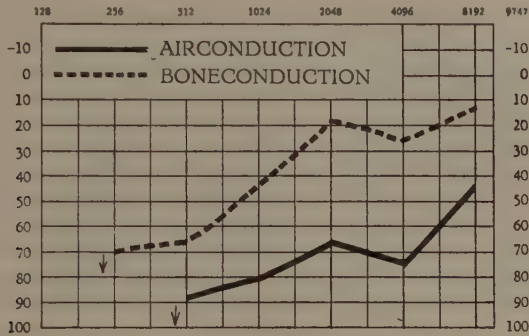


Fig. 1.

As a matter of course the bone conduction in this audiogram has suffered much less than the air conduction.

More often, however, in otosclerosis a deafness for all tones was found. *A deafness for the high tones alone was never present.* This is in contradiction to the forms of deafness which will be discussed presently.

As an illustration the average audiogram of the second group of deafness of 9 otosclerotic ears (4 patients with a bilateral, 1 patient with an unilateral process) is given.

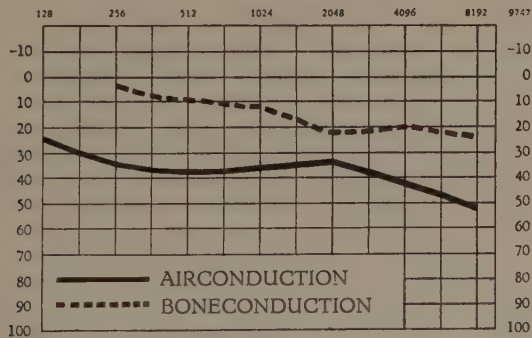


Fig. 2.

As average audiogram we mean those audiograms, which, in a series of cases of one and the same affection, give for each tone the arithmetical-average threshold, on which the tone can just be heard. It is therefore the typical audiogram for that special affection, resulting in this way.

In cases of otitis media the ossicular chain, the tympanic membrane and the mucous membrane of the middle ear are all involved. Without further discussion of this process we give here some audiograms for comparison (2). These are the average audiograms of 6 ears with acute catarrhal otitis media (fig. 3a) and of 12 ears with chronic scar-like otitis media, or with an otitis media healed with a perforation (fig. 3b). Each of the original audiograms showed an increase (as far as the air conduction is concerned) of the threshold value for *all* tones.

All these forms of transmission deafness gave only slightly differing audiograms. They show a deafness for alle tones, except in one case, i.e. in otosclerosis where sometimes a bass-deafness is found.

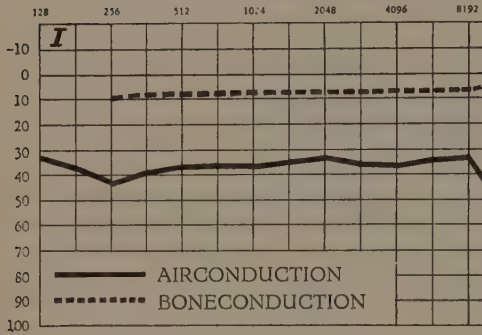


Fig. 3a.

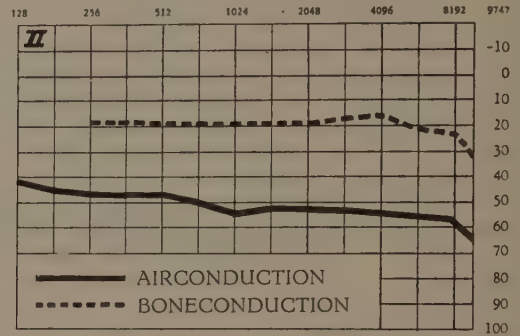


Fig. 3b.

One large group of transmission-deafness still waits for discussion: a deafness which is localised exclusively or mainly in the tympanic membrane.

When under pressure, the tympanic membrane is impeded in its movements. This is the case with the so-called retracted tympanic membrane when by insufficient passage of the tuba Eustachii the resorbed air in the tympanic cavity is not supplied, thus causing a crushing of the tympanic membrane by the tension of the open air. This condition especially develops in cases of tubal catarrh. This decreased pressure in the cavum tympani in cases of tubal catarrh can be incontestably demonstrated with the pneumophone of V. DISHOECK.

It was already known since a long time that in cases of tubal catarrh the high tones were mainly impaired; this fact, however, had been forgotten and the following conception had become an axioma: in such cases of transmission deafness it is especially the bass-side which is impaired. CROWE, BAYLOR and GUILD (l.c.) on the ground of their audiometric investigations, again pointed to the incorrectness of this rule. Formerly we also examined 8 ears with symptoms of slight tubal catarrh. The audiograms showed only a loss of hearing for the high tones (fig. 4).

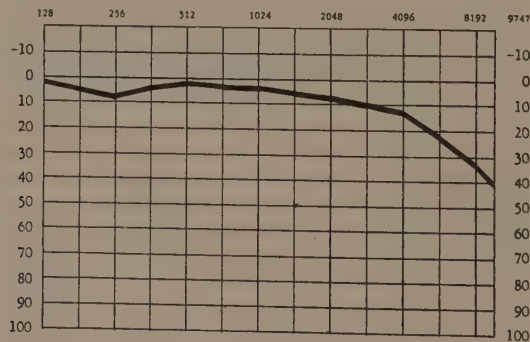


Fig. 4.

In these cases of catarrh inflammatory moments must always be taken into consideration. This also (although less pronounced) is the case in the following group.

We ourselves examined 15 patients (children and girls of the age of 8—25 years)

belonging to this group, all having adenoid growths. In general these patients had a retracted tympanic membrane (most often with a normal hearing for the whispering voice!) which was undoubtedly caused by occlusion of the tuba by the adenoid. Of these 30 ears an audiogram was made one day before and one month after the adenotomy.

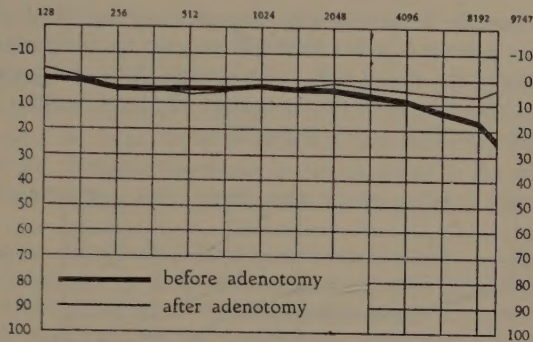


Fig. 5.

From these average audiograms it appears that again only the hearing for high tones is disturbed. That this disturbance was due to the adenoid, follows from the fact that it completely disappeared after adenotomy.

In these cases of increased pressure in the tympanic membrane, we not only have to take into account the abnormal pressure on the tympanic membrane, but also an eventual influence of the changed position of the ossicles. The excursion of the malleus handle inwards cannot be neglected.

However, some experiments exist which point to the fact that it is mainly the disturbance of the tympanic membrane by which this discant deafness is caused.

LÜSCHER (3) investigated the influence of an artificial loading of the tympanic membrane upon the auditory acuteness. He applied several quantities of mercury and water upon the tympanic membrane, especially upon the lower part, the pars tensa. It appeared that this gave rise to a discant deafness. From the comparison of the results of the experiments with water and with mercury he also deduced that it was not only or mainly the weight of the applied fluid which caused the disturbances, but that the intensity of the disturbance depended especially from the degree in which the tympanic membrane was covered. Hence it was especially the disturbed vibration of the tympanic membrane which was responsible for the deafness.

Another support for this conception is found in the communication of T. MIKI (4). He saw that pledgets of wad which were steeped in paraffin and applied upon the tympanic membrane, caused chiefly a discant deafness. He made this experiment after having observed that a mass of cerumen normally caused a decrease of the hearing acuteness, but, when they are adhered to the tympanic membrane cause an extra impairment of the high tones.

Before and after syringing a mass of cerumen in 4 ears, we made an audiogram. It appeared that the loss of hearing by occlusion is complete and about 20 db. The increase of hearing after syringing was for the 128 Hz tone: 16 db; for the 256 Hz tone: 14 db; for the 512 Hz tone: 14 db; for the 1024 Hz tone: 14 db; for the 2048 Hz tone: 19 db; for the 4096 Hz tone: 20 db; for the 8192 Hz tone: 19 db and for the 9747 Hz tone: 22 db.

We asked ourselves whether it were possible to find cases in the otological clinic in which (as far as the transmission mechanism is concerned) simply and solely the tympanic membrane was affected.

These were found among the cases of otitis externa. As the epithelium of the auditory canal proceeds on the tympanic membrane it is very acceptable that a process of this

epithelium extends upon the tympanic membrane. Often the tympanic membrane is also dull and thickened in otitis externa.

Up to now a systematical audiometric investigation of the auditory acuteness in cases of otitis externa was absent: by the above mentioned observations it gets a certain importance.

We were able to examine 4 patients with uncomplicated unilateral otitis externa. The 4 healthy ears of these patients supplied favorable material for comparison¹⁾. Fig. 6 gives the average audiogram of the 4 ears with external otitis. All audiograms separately show a discant deafness. With thin lines the audiogram of the 4 healthy ears is given. It appears that especially the discant side is impaired.

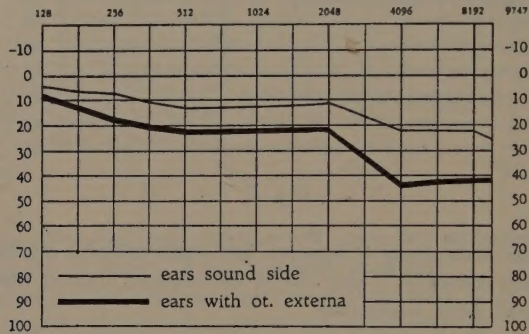


Fig. 6.

Furthermore we examined 2 persons with a bilateral external otitis. Fig. 8 gives the average audiogram of the 4 ears.

Here, still more than in fig. 6, the disturbances at the discant side are pronounced.

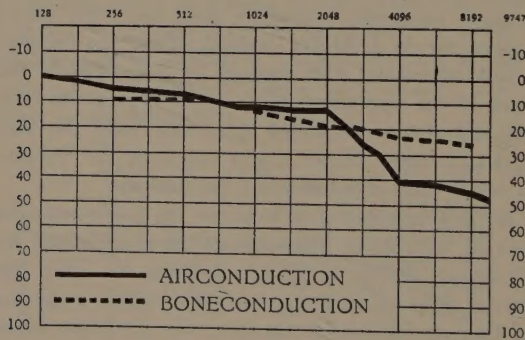


Fig. 7.

The examination of the purest localisation of the disturbed transmission of sound in the tympanic membrane, especially in cases of external otitis, made it probable that the tympanic membrane is of *much importance for the transmission of the high tones*.

This conception, i.e. that for the transmission of the high tones the tympanic membrane is of great importance, is supported by the fact that otosclerosis is the single affection

¹⁾ The 4 healthy ears also show a slight loss of hearing at the discant side. Probably, although not found, here in some cases a slight otitis externa may have been present.

in which a bass-deafness with conservation of the hearing for high tones is found. It is sure that in otosclerosis the tympanic membrane is not affected and can fulfil its function normally.

The investigation of LÜSCHER makes it probable that it is especially the pars tensa of the tympanic membrane which is of importance for this transmission ¹⁾.

Summary.

From determinations of the hearing acuteness in several cases of transmission deafness it appears that in morbid processes with involvement of the tympanic membrane always the hearing acuteness for high tones is impaired. On the other hand otosclerosis (an affection in which the tympanic membrane is not involved) is the sole disease in which in some cases a bass-deafness with conservation of the auditory acuteness for high tones can be found.

In cases of external otitis where (as far as the transmission mechanism is concerned) only the tympanic membrane is affected, a decrease of the hearing is found mainly for high tones. This forms a strong argument for the above mentioned conception that the tympanic membrane is very important for the transmission of high tones. The experiments of LÜSCHER should point to the part the pars tensa plays in this transmission.

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3. E. LÜSCHER, Acta Otolaryngol., **27**, 250 (1939).
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¹⁾ A more extensive experimental study about the function of the tympanic membrane, from H. A. E. V. DISHOECK and the author, will appear shortly.

